Chapter 4

POSITION ANALYSIS

Theory is the distilled essence of practice
RANKINE

4.0 INTRODUCTION

Once a tentative mechanism design has been **synthesized**, it must then be **analyzed**. A principal goal of kinematic analysis is to determine the accelerations of all the moving parts in the assembly. **Dynamic forces** are proportional to acceleration, from Newton's second law. We need to know the dynamic forces in order to calculate the **stresses** in the components. The design engineer must ensure that the proposed mechanism or machine will not fail under its operating conditions. Thus the stresses in the materials must be kept well below allowable levels. To calculate the stresses, we need to know the static and dynamic forces on the parts. To calculate the dynamic forces, we need to know the **accelerations**. In order to calculate the accelerations, we must first find the **positions** of all the links or elements in the mechanism for each increment of input motion, and then differentiate the position equations versus time to find **velocities**, and then differentiate again to obtain the expressions for acceleration. For example, in a simple Grashof fourbar linkage, we would probably want to calculate the positions, velocities, and accelerations of the output links (coupler and rocker) for perhaps every two degrees (180 positions) of input crank position for one revolution of the crank.

This can be done by any of several methods. We could use a **graphical approach** to determine the position, velocity, and acceleration of the output links for all 180 positions of interest, or we could **derive the general equations** of motion for any position, differentiate for velocity and acceleration, and then solve these **analytical expressions** for our

180 (or more) crank locations. A computer will make this latter task much more palatable. If we choose to use the graphical approach to analysis, we will have to do an independent graphical solution for each of the positions of interest. None of the information obtained graphically for the first position will be applicable to the second position or to any others. In contrast, once the analytical solution is derived for a particular mechanism, it can be quickly solved (with a computer) for all positions. If you want information for more than 180 positions, it only means you will have to wait longer for the computer to generate those data. The derived equations are the same. So, have another cup of coffee while the computer crunches the numbers! In this chapter, we will present and derive analytical solutions to the position analysis problem for various planar mechanisms. We will also discuss graphical solutions which are useful for checking your analytical results. In Chapters 6 and 7 we will do the same for velocity and acceleration analysis of planar mechanisms.

It is interesting to note that **graphical position analysis** of linkages is a truly trivial exercise, while the algebraic approach to position analysis is much more complicated. If you can draw the linkage to scale, you have then solved the position analysis problem graphically. It only remains to measure the link angles on the scale drawing to protractor accuracy. But the converse is true for velocity and especially for acceleration analysis. Analytical solutions for these are less complicated to derive than is the analytical position solution. However, graphical velocity and acceleration analysis becomes quite complex and difficult. Moreover, the graphical vector diagrams must be redone *de novo* (meaning literally *from new*) for each of the linkage positions of interest. This is a very tedious exercise and was the only practical method available in the days *B.C.* (*Before Computer*), not so long ago. The proliferation of inexpensive microcomputers in recent years has truly revolutionized the practice of engineering. As a graduate engineer, you will never be far from a computer of sufficient power to solve this type of problem and may even

Geez Joe, - now I wish I took that programming course!

have one in your pocket. Thus, in this text we will emphasize analytical solutions which are easily solved with a microcomputer. The computer programs provided with this text use the same analytical techniques as derived in the text.

4.1 COORDINATE SYSTEMS

Coordinate systems and reference frames exist for the convenience of the engineer who defines them. In the next chapters we will provide our systems with multiple coordinate systems as needed, to aid in understanding and solving the problem. We will denote one of these as the *global* or *absolute* coordinate system, and the others will be *local* coordinate systems within the global framework. The global system is often taken to be attached to Mother Earth, though it could as well be attached to another ground plane such as the frame of an automobile. If our goal is to analyze the motion of a windshield wiper blade, we may not care to include the gross motion of the automobile in the analysis. In that case a global coordinate system (GCS—denoted as X,Y) attached to the car would be useful, and we could consider it to be an **absolute** coordinate system. Even if we use the earth as an absolute reference frame, we must realize that it is not stationary either, and as such is not very useful as a reference frame for a space probe. Though we will speak of absolute positions, velocities, and accelerations, keep in mind that ultimately, until we discover some stationary point in the universe, all motions are really relative. The term **inertial reference frame** is used to denote *a system which itself has no acceleration.* All angles in this text will be measured according to the *right-hand rule*. That is, **counterclockwise angles**, angular velocities, and angular accelerations *are positive in sign.*

Local coordinate systems are typically attached to a link at some point of interest. This might be a pin joint, a center of gravity, or a line of centers of a link. These local coordinate systems may be either rotating or nonrotating as we desire. If we want to measure the angle of a link as it rotates in the global system, we probably will want to attach a local nonrotating coordinate system (LNCS—denoted as x, y) to some point on the link (say a pin joint). This nonrotating system will move with its origin on the link but remains always parallel to the global system. If we want to measure some parameters within a link, independent of its rotation, then we will want to construct a local rotating coordinate system (LRCS—denoted as x', y') along some line on the link. This system will both move and rotate with the link in the global system. Most often we will need to have both types of local coordinate systems (LNCS and LRCS) on our moving links to do a complete analysis. Obviously we must define the angles and/or positions of these moving, local coordinate systems in the global system at all positions of interest.

* Note that a two-argument arctangent function must be used to obtain angles in all four quadrants. The single-argument arctangent function found in most calculators and computer programming languages returns angle values in only the first and fourth quadrants. You can calculate your own two-argument arctangent function very easily by testing the sign of the x component of the arguments and, if x is minus, adding π radians or 180° to the result obtained from the available single-argument arctangent function.

For example (in Fortran):

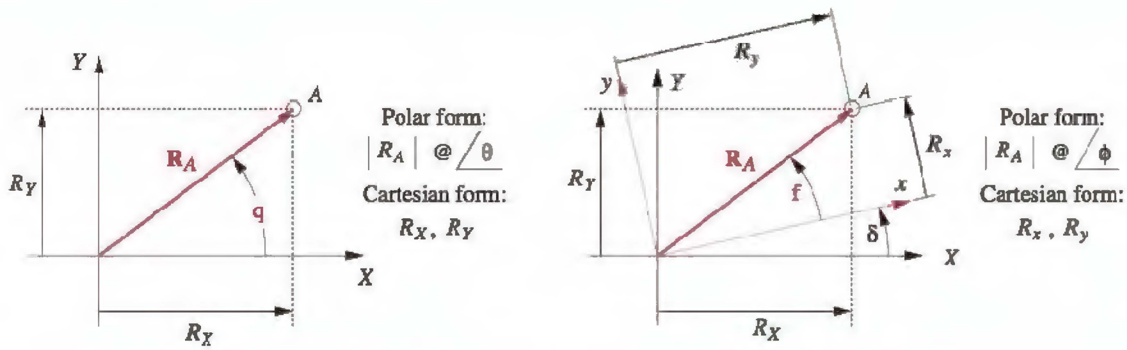
```
FUNCTION Atan2( x, y )
IF x <> 0 THEN Q = y / x
Temp = ATAN(Q)
IF x < 0 THEN
  Atan2 = Temp + 3.14159
ELSE
  Atan2 = Temp
END IF
RETURN
END
```

The above code assumes that the language used has a built-in single-argument arctangent function called ATAN(x) which returns an angle between $\pm\pi/2$ radians when given a signed argument representing the value of the tangent of that angle.

4.2 POSITION AND DISPLACEMENT

Position

The **position** of a point in the plane can be defined by the use of a **position vector** as shown in Figure 4-1. The choice of **reference axes** is arbitrary and is selected to suit the observer. Figure 4-1a shows a point in the plane defined in a global coordinate system and Figure 4-1b shows the same point defined in a local coordinate system with its origin coincident with the global system A two-dimensional vector has two attributes, which can be expressed in either *polar* or *cartesian* coordinates. The **polar form** provides the



(*a*) Global coordinate system *XY* (*b*) Local coordinate system *xy*

FIGURE 4-1

A position vector in the plane - expressed in both global and local coordinates

4

magnitude and the angle of the vector. The **cartesian form** provides the *X* and *Y* components of the vector. Each form is directly convertible into the other by*

the Pythagorean theorem:

$$R_A = \sqrt{R_X^2 + R_Y^2}$$

and trigonometry: (4.0a)

$$\theta = \arctan\left(\frac{R_Y}{R_X}\right)$$

Equations 4.0a are shown in global coordinates but could as well be expressed in local coordinates.

Coordinate Transformation

It is often necessary to transform the coordinates of a point defined in one system to coordinates in another. If the system's origins are coincident as shown in Figure 4-1b and the required transformation is a rotation, it can be expressed in terms of the original coordinates and the signed angle δ between the coordinate systems. If the position of point *A* in Figure 4-1b is expressed in the local *xy* system as R_x, R_y, and it is desired to transform its coordinates to R_X, R_Y in the global *XY* system, the equations are:

$$\begin{aligned} R_X &= R_x \cos\delta - R_y \sin\delta \\ R_Y &= R_x \sin\delta + R_y \cos\delta \end{aligned} \tag{4.0b}$$

Displacement

Displacement of a point is the change in its position and can be defined as *the straight-line distance between the initial and final position of a point which has moved in the reference frame.* Note that displacement is not necessarily the same as the path length which the point may have traveled to get from its initial to final position. Figure 4-2a shows a

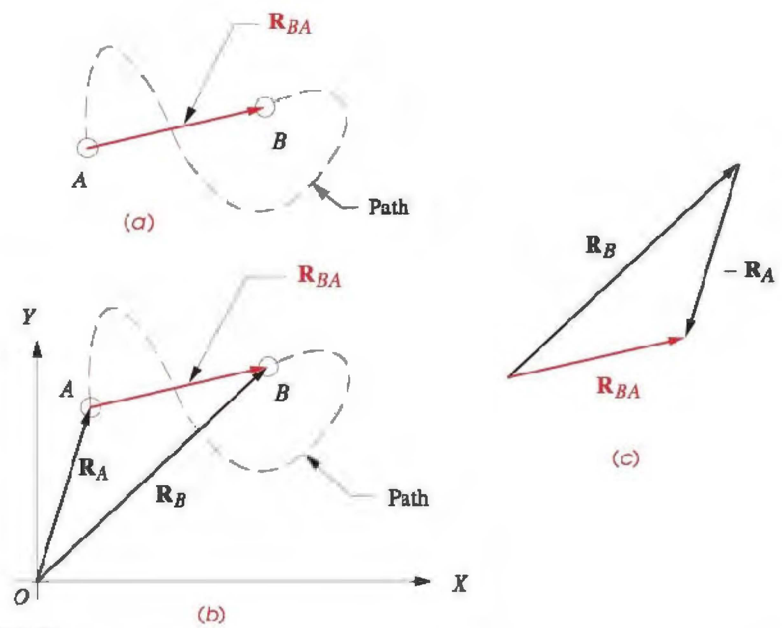


FIGURE 4-2

Position difference and relative position

point in two positions, A and B. The curved line depicts the path along which the point traveled. The position vector $\mathbf{R}_{BA}$ defines the displacement of the point B with respect to point A. Figure 4-2b defines this situation more rigorously and with respect to a reference frame XY. The notation $\mathbf{R}$ will be used to denote a position vector. The vectors $\mathbf{R}_A$ and $\mathbf{R}_B$ define, respectively, the absolute positions of points A and B with respect to this *global XY* reference frame. The vector $\mathbf{R}_{BA}$ denotes the difference in position, or the *displacement*, between A and B. This can be expressed as the *position difference equation:*

$$\mathbf{R}_{BA} = \mathbf{R}_B - \mathbf{R}_A \tag{4.1a}$$

This expression is read: *The position of B with respect to A is equal to the (absolute) position of B minus the (absolute) position of A*, where *absolute* means with respect to the origin of the *global* reference frame. This expression could also be written as:

$$\mathbf{R}_{BA} = \mathbf{R}_{BO} - \mathbf{R}_{AO} \tag{4.1b}$$

with the second subscript O denoting the origin of the XY reference frame. When a position vector is rooted at the origin of the reference frame, it is customary to omit the second subscript. It is understood, in its absence, to be the origin. Also, a vector referred to the origin, such as $\mathbf{R}_A$, is often called an absolute vector. This means that it is taken with respect to a reference frame which is assumed to be stationary, e.g., *the ground*. It is important to realize, however, that the ground is usually also in motion in some larger frame of reference. Figure 4-2c shows a graphical solution to equations 4.1.

In our example of Figure 4-2, we have tacitly assumed so far that this point, which is first located at A and later at B, is, in fact, the same particle, moving within the reference frame. It could be, for example, one automobile moving along the road from A to B. With that assumption, it is conventional to refer to the vector $\mathbf{R}_{BA}$ as a **position difference**. There is, however, another situation which leads to the same diagram and equation but needs a different name. Assume now that points A and B in Figure 4-2b represent not the same particle but two independent particles moving in the same reference frame, as perhaps two automobiles traveling on the same road. The vector equations 4.1 and the diagram in Figure 4-2b still are valid, but we now refer to $\mathbf{R}_{BA}$ as a **relative position**, or **apparent position**. We will use the *relative position* term here. A more formal way to distinguish between these two cases is as follows:

CASE 1: *One body in two successive positions => **position difference***

CASE 2: *Two bodies simultaneously in separate positions => **relative position***

This may seem a rather fine point to distinguish, but the distinction will prove useful, and the reasons for it more clear, when we analyze velocities and accelerations, especially when we encounter (Case 2 type) situations in which the two bodies occupy the same position at the same time but have different motions.

4.3 TRANSLATION, ROTATION, AND COMPLEX MOTION

So far we have been dealing with a particle, or point, in plane motion. It is more interesting to consider the motion of a **rigid body**, or link, which involves both the position of a point on the link and the orientation of a line on the link, sometimes called the **POSE** of the link. Figure 4-3a (p. 180) shows a link AB denoted by a position vector $\mathbf{R}_{BA}$. An axis system has been set up at the root of the vector, at point A, for convenience.

Translation

Figure 4-3b shows link AB moved to a new position $A'B'$ by translation through the displacement AA' or BB' which are equal, i.e., $\mathbf{R}_{A'A} = \mathbf{R}_{B'B}$.

A definition of *translation* is:

All points on the body have the same displacement.

As a result the link retains its angular orientation. Note that the translation need not be along a straight path. The curved lines from A to A' and B to B' are the **curvilinear translation** path of the link. There is no rotation of the link if these paths are parallel. If the path happens to be straight, then it will be the special case of **rectilinear translation**, and the path and the displacement will be the same.

Rotation

Figure 4-3c shows the same link AB moved from its original position at the origin by rotation through an angle. Point A remains at the origin, but B moves through the position difference vector $\mathbf{R}_{B'B} = \mathbf{R}_{B'A} - \mathbf{R}_{BA}$.

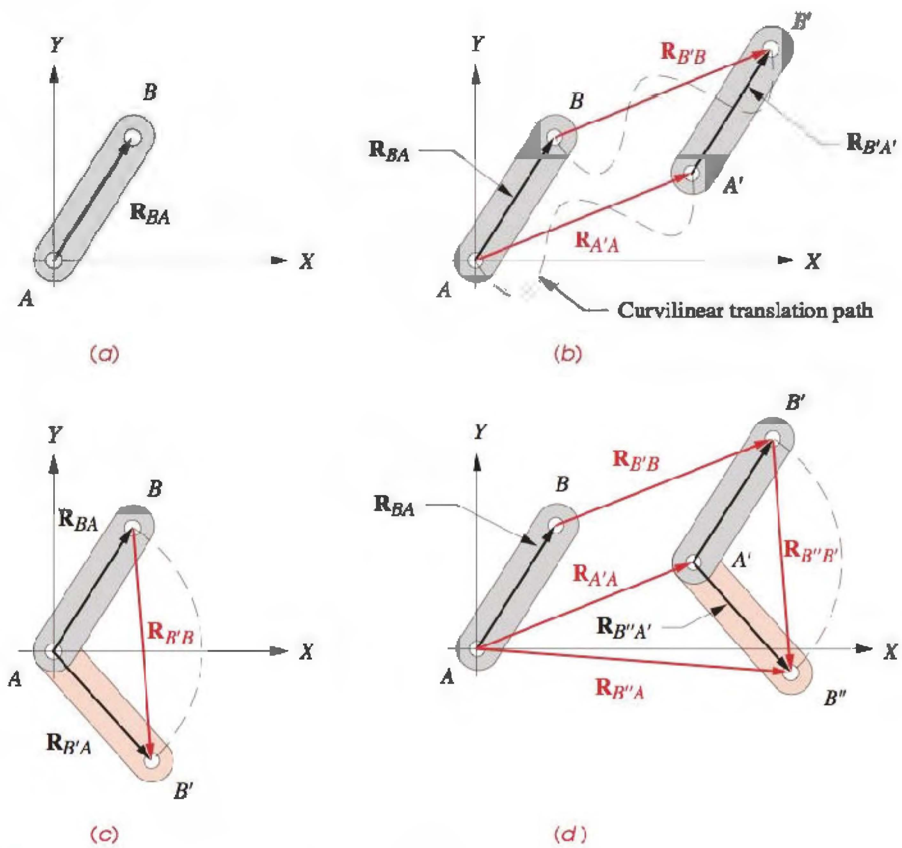


FIGURE 4-3

Translation, rotation, and complex motion

A definition of *rotation* is:

Different points in the body undergo different displacements and thus there is a displacement difference between any two points chosen.

The link now changes its angular orientation in the reference frame, and all points have different displacements.

Complex Motion

The general case of **complex motion** is the sum of the translation and rotation components. Figure 4-3d shows the same link moved through both a translation and a rotation. Note that the order in which these two components are added is immaterial. The resulting complex displacement will be the same whether you first rotate and then translate or vice versa. This is so because the two factors are independent. The total complex displacement of point B is defined by the following expression:

Total displacement = translation component + rotation component

$$\mathbf{R}_{B''B} = \mathbf{R}_{B'B} + \mathbf{R}_{B''B'} \tag{4.1c}$$

The new absolute position of point B referred to the origin at A is:

$$\mathbf{R}_{B''A} = \mathbf{R}_{A'A} + \mathbf{R}_{B''A'} \tag{4.1d}$$

Note that the above two formulas are merely applications of the position difference equation 4.1a (p. 178). See also Section 2.2 (p. 31) for definitions and discussion of *rotation, translation,* and *complex motion.* These motion states can be expressed as the following theorems.

4

Theorems

Euler's theorem:

The general displacement of a rigid body with one point fixed is a rotation about some axis.

This applies to pure rotation as defined above and in Section 2.2 (p. 31). Chasles (1793-1880) provided a corollary to Euler's theorem now known as:

Chasles' theorem:[6] *

Any displacement of a rigid body is equivalent to the sum of a translation of any one point on that body and a rotation of the body about an axis through that point.

This describes complex motion as defined above and in Section 2.2. Note that equation 4.1c is an expression of Chasles' theorem.

4.4 GRAPHICAL POSITION ANALYSIS OF LINKAGES

For any one-*DOF* linkage, such as a fourbar, only one parameter is needed to completely define the positions of all the links. The parameter usually chosen is the angle of the input link. This is shown as θ_2 in Figure 4-4 (p. 182). We want to find θ_3 and θ_4. The link lengths are known. Note that we will consistently number the ground link as 1 and the driver link as 2 in these examples.

The graphical analysis of this problem is trivial and can be done using only high-school geometry. If we draw the linkage carefully to scale with rule, compass, and protractor in a particular position (given θ_2), then it is only necessary to measure the angles of links 3 and 4 with the protractor. Note that all link angles are measured from a positive X axis. In Figure 4-4, a *local xy* axis system, parallel to the *global XY* system, has been created at point A to measure θ_3. The accuracy of this graphical solution will be limited by our care and drafting ability and by the crudity of the protractor used. Nevertheless, a very rapid approximate solution can be found for any one position.

Figure 4-5 (p. 182) shows the construction of the graphical position solution. The four link lengths a, b, c, d and the angle θ_2 of the input link are given. First, the ground link (1) and the input link (2) are drawn to a convenient scale such that they intersect at the origin O_2 of the global XY coordinate system with link 2 placed at the input angle θ_2. Link 1 is drawn along the X axis for convenience. The compass is set to the scaled length of link 3, and an arc of that radius is swung about the end of link 2 (point A). Then the

* Ceccarelli[7] points out that Chasles' theorem (Paris, 1830) was put forth earlier (Naples, 1763) by Mozzi[8] but the latter's work was apparently unknown or ignored in the rest of Europe, and the theorem became associated with Chasles' name.

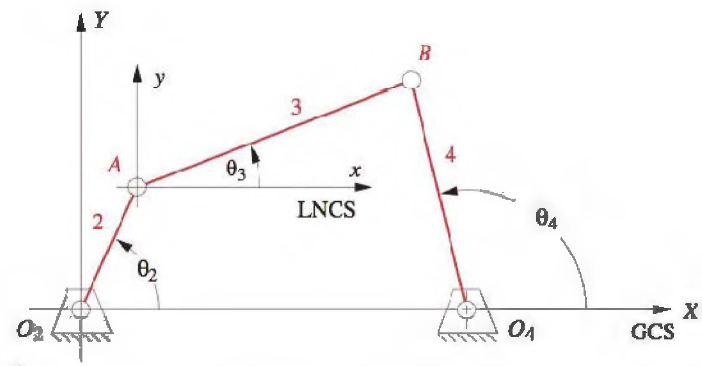


FIGURE 4-4

Measurement of angles in the fourbar linkage

compass is set to the scaled length of link 4, and a second arc is swung about the end of link 1 (point O_4). These two arcs will have two intersections at B and B' that define the two solutions to the position problem for a fourbar linkage which can be assembled in two configurations, called circuits, labeled open and crossed in Figure 4-5. Circuits in linkages will be discussed in a later section.

The angles of links 3 and 4 can be measured with a protractor. One circuit has angles θ_3 and θ_4, the other $\theta_{3'}$ and $\theta_{4'}$. A graphical solution is only valid for the particular value of input angle used. For each additional position analysis we must completely redraw the linkage. This can become burdensome if we need a complete analysis at every 1- or 2-degree increment of θ_2. In that case we will be better off to derive an analytical solution for θ_3 and θ_4 that can be solved by computer.

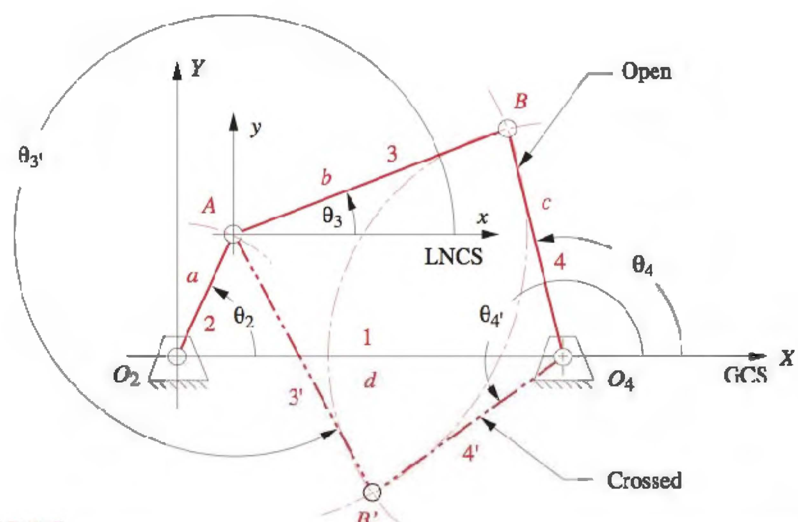


FIGURE 4-5

Graphical position solution to the open and crossed configurations of the fourbar linkage

4.5 ALGEBRAIC POSITION ANALYSIS OF LINKAGES

The same procedure that was used in Figure 4-5 to solve geometrically for the intersections B and B' and angles of links 3 and 4 can be encoded into an algebraic algorithm. The coordinates of point A are found from

$$A_x = a\cos\theta_2 \qquad A_y = a\sin\theta_2 \tag{4.2a}$$

The coordinates of point B are found using the equations of circles about A and O_4.

$$b^2 = (B_x - A_x)^2 + (B_y - A_y)^2 \tag{4.2b}$$

$$c^2 = (B_x - d)^2 + B_y^2 \tag{4.2c}$$

which provide a pair of simultaneous equations in B_x and B_y.

Subtracting equation 4.2c from 4.2b gives an expression for B_x.

$$B_x = \frac{a^2 - b^2 + c^2 - d^2}{2(A_x - d)} - \frac{2A_yB_y}{2(A_x - d)} = S - \frac{2A_yB_y}{2(A_x - d)} \tag{4.2d}$$

Substituting equation 4.2d into 4.2c gives a quadratic equation in B_y which has two solutions corresponding to those in Figure 4-5.

$$B_y^2 + \left(S - \frac{A_yB_y}{A_x - d} - d\right)^2 - c^2 = 0 \tag{4.2e}$$

This can be solved with the familiar expression for the roots of a quadratic equation,

$$B_y = \frac{-Q \pm \sqrt{Q^2 - 4PR}}{2P} \tag{4.2f}$$

where:

$$P = \frac{A_y^2}{(A_x - d)^2} + 1 \qquad Q = \frac{2A_y(d - S)}{A_x - d}$$

$$R = (d - S)^2 - c^2 \qquad S = \frac{a^2 - b^2 + c^2 - d^2}{2(A_x - d)}$$

Note that the solutions to this equation set can be real or imaginary. If the latter, it indicates that the links cannot connect at the given input angle or at all. Once the two values of B_y are found (if real), they can be substituted into equation 4.2d to find their corresponding x components. The link angles for this position can then be found from

$$\theta_3 = \tan^{-1}\left(\frac{B_y - A_y}{B_x - A_x}\right) \qquad \theta_4 = \tan^{-1}\left(\frac{B_y}{B_x - d}\right) \tag{4.2g}$$

A two-argument arctangent function must be used to solve equations 4.2g since the angles can be in any quadrant. Equations 4.2 can be encoded in any computer language or equation solver, and the value of θ_2 varied over the linkage's usable range to find all corresponding values of the other two link angles.

Vector Loop Representation of Linkages

4

An alternate approach to linkage position analysis creates a vector loop (or loops) around the linkage as first proposed by Raven.[9] This approach offers some advantages in the synthesis of linkages which will be addressed in Chapter 5. The links are represented as **position vectors**. Figure 4-6 shows the same fourbar linkage as in Figure 4-4 (p. 182), but the links are now drawn as position vectors that form a vector loop. This loop closes on itself, making the sum of the vectors around the loop zero. The lengths of the vectors are the link lengths, which are known. The current linkage position is defined by the input angle θ_2 as it is a one-*DOF* mechanism. We want to solve for the unknown angles θ_3 and θ_4. To do so we need a convenient notation to represent the vectors.

Complex Numbers as Vectors

There are many ways to represent vectors. They may be defined in **polar coordinates**, by their *magnitude* and *angle*, or in **cartesian coordinates** as x and y components. These forms are of course easily convertible from one to the other using equations 4.0a. The position vectors in Figure 4-6 can be represented as any of these expressions:

Polar form	Cartesian form	
$R@\angle\theta$	$r\cos\theta\,\hat{\mathbf{i}}+r\sin\theta\,\hat{\mathbf{j}}$	(4.3a)
$re^{j\theta}$	$r\cos\theta+jr\sin\theta$	(4.3b)

Equation 4.3a uses **unit vectors** to represent the x and y vector component directions in the cartesian form. Figure 4-7 shows the unit vector notation for a position vector. Equation 4.3b uses **complex number notation** wherein the X direction component is called the *real portion* and the Y direction component is called the *imaginary portion*. This unfortunate term *imaginary* comes about because of the use of the notation j to represent the square root of minus one, which of course cannot be evaluated numerically.

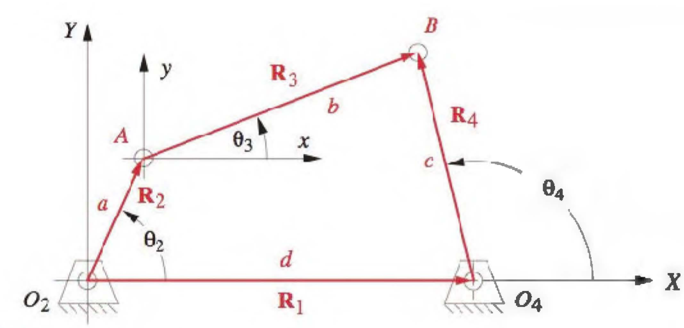


FIGURE 4-6

Position vector loop for a fourbar linkage

However, this *imaginary* number is used in a **complex number** as an **operator,** *not as a value.* Figure 4-8a (p. 186) shows the **complex plane** in which the *real* axis represents the *X*-directed component of the vector in the plane, and the *imaginary* axis represents the *Y*-directed component of the same vector. So, any term in a complex number which has no *j* operator is an *x* component, and a *j* indicates a *y* component.

Note in Figure 4-8b (p. 186) that each multiplication of the vector $\mathbf{R}_A$ by the operator *j* results in a *counterclockwise rotation* of the vector through 90 degrees. The vector $\mathbf{R}_B = j\mathbf{R}_A$ is directed along the *positive imaginary* or *j* axis. The vector $\mathbf{R}_C = j^2\mathbf{R}_A$ is directed along the *negative real* axis because $j^2 = -1$ and thus $\mathbf{R}_C = -\mathbf{R}_A$. In similar fashion, $\mathbf{R}_D = j^3\mathbf{R}_A = -j\mathbf{R}_A$ and this component is directed along the *negative j axis.*

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One advantage of using this complex number notation to represent planar vectors comes from the **Euler identity:**

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{4.4a}$$

Any two-dimensional vector can be represented by the compact polar notation on the left side of equation 4.4a. There is no easier function to differentiate or integrate, since it is its own derivative:

$$\frac{de^{j\theta}}{d\theta} = je^{j\theta} \tag{4.4b}$$

We will use this **complex number notation** for vectors to develop and derive the equations for position, velocity, and acceleration of linkages.

The Vector Loop Equation for a Fourbar Linkage

The directions of the position vectors in Figure 4-6 are chosen so as to define their angles where we desire them to be measured. By definition, ***the angle of a vector is always measured at its root, not at its head.*** We would like angle θ_4 to be measured at the fixed pivot O_4, so vector $\mathbf{R}_4$ is arranged to have its root at that point. We would like to measure angle θ_3 at the point where links 2 and 3 join, so vector $\mathbf{R}_3$ is rooted there. A similar logic dictates the arrangement of vectors $\mathbf{R}_1$ and $\mathbf{R}_2$. Note that the *X* (*real*) axis is taken for convenience along link 1 and the origin of the global coordinate system is taken at point

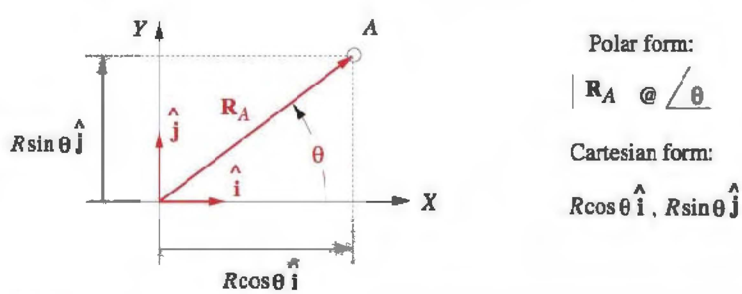
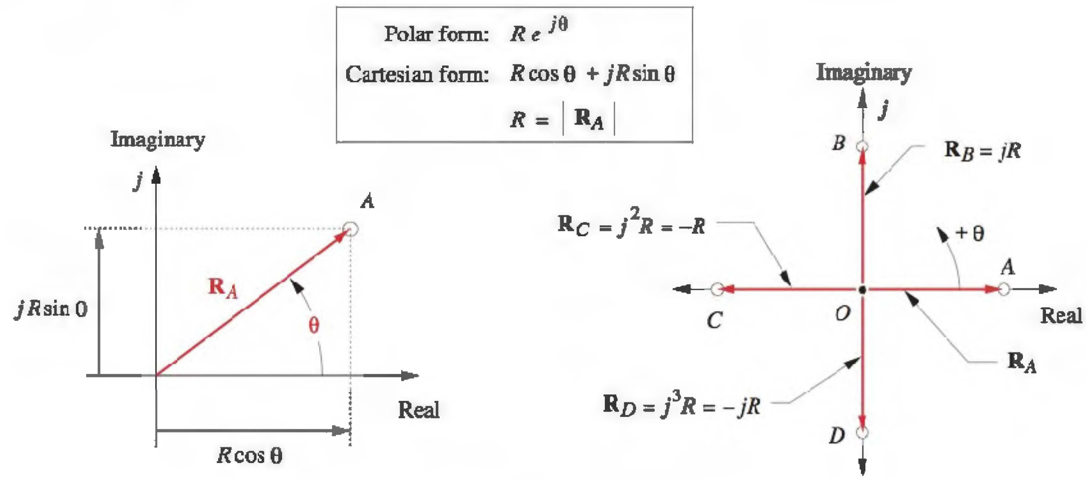


FIGURE 4-7
Unit vector notation for position vectors



(*a*) Complex number representation of a position vector

(*b*) Vector rotations in the complex plane

FIGURE 4-8

Complex number representation of vectors in the plane

O_2, the root of the input link vector $\mathbf{R}_2$. These choices of vector directions and senses, as indicated by their arrowheads, lead to this vector loop equation:

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.5a}$$

An **alternate notation** for these position vectors is to use the labels of the points at the vector **tips and roots** (*in that order*) as subscripts. The second subscript is conventionally omitted if it is the origin of the global coordinate system (point O_2):

$$\mathbf{R}_A + \mathbf{R}_{BA} - \mathbf{R}_{BO_4} - \mathbf{R}_{O_4} = 0 \tag{4.5b}$$

Next, we substitute the complex number notation for each position vector. To simplify the notation and minimize the use of subscripts, we will denote the scalar lengths of the four links as *a*, *b*, *c*, and *d*. These are so labeled in Figure 4-6 (p. 184). The equation then becomes:

$$a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \tag{4.5c}$$

These are three forms of the same vector equation, and as such can be solved for two unknowns. There are four variables in this equation, namely the four link angles. The link lengths are all constant in this particular linkage. Also, the value of the angle of link 1 is fixed (at zero) since this is the ground link. The *independent variable* is θ_2 which we will control with a motor or other driver device. That leaves the angles of link 3 and 4 to be found. We need algebraic expressions which define θ_3 and θ_4 as functions only of the constant link lengths and the one input angle, θ_2. These expressions will be of the form:

$$\theta_3 = f\{a, b, c, d, \theta_2\}$$
$$\theta_4 = g\{a, b, c, d, \theta_2\} \tag{4.5d}$$

To solve the polar form, vector equation 4.5c, we must substitute the *Euler equivalents* (equation 4.4a, p. 185) for the $e^{j\theta}$ terms, and then separate the resulting cartesian form vector equation into two scalar equations which can be solved simultaneously for θ_3 and θ_4. Substituting equation 4.4a into equation 4.5c:

$$a(\cos\theta_2 + j\sin\theta_2) + b(\cos\theta_3 + j\sin\theta_3) - c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0 \tag{4.5e}$$

This equation can now be separated into its real and imaginary parts and each set to zero.

real part (x component):

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so: (4.6a)

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$$

imaginary part (y component):

$$ja\sin\theta_2 + jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the j's divide out, so: (4.6b)

$$a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 = 0$$

The scalar equations 4.6a and 4.6b can now be solved simultaneously for θ_3 and θ_4. To solve this set of two simultaneous trigonometric equations is straightforward but tedious. Some substitution of trigonometric identities will simplify the expressions. The first step is to rewrite equations 4.6a and 4.6b so as to isolate one of the two unknowns on the left side. We will isolate θ_3 and solve for θ_4 in this example.

$$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d \tag{4.6c}$$
$$b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4 \tag{4.6d}$$

Now square both sides of equations 4.6c and 4.6d and add them:

$$b^2\left(\sin^2\theta_3 + \cos^2\theta_3\right) = \left(-a\sin\theta_2 + c\sin\theta_4\right)^2 + \left(-a\cos\theta_2 + c\cos\theta_4 + d\right)^2 \tag{4.7a}$$

Note that the quantity in parentheses on the left side is equal to 1, eliminating θ_3 from the equation, leaving only θ_4 which can now be solved for.

$$b^2 = \left(-a\sin\theta_2 + c\sin\theta_4\right)^2 + \left(-a\cos\theta_2 + c\cos\theta_4 + d\right)^2 \tag{4.7b}$$

Expand this expression and collect terms.

$$b^2 = a^2 + c^2 + d^2 - 2ad\cos\theta_2 + 2cd\cos\theta_4 - 2ac\left(\sin\theta_2\sin\theta_4 + \cos\theta_2\cos\theta_4\right) \tag{4.7c}$$

Divide through by $2ac$ and rearrange to get:

4

$$\frac{d}{a}\cos\theta_4 - \frac{d}{c}\cos\theta_2 + \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \sin\theta_2\sin\theta_4 + \cos\theta_2\cos\theta_4 \tag{4.7d}$$

To further simplify this expression, the constants K_1, K_2, and K_3 are defined in terms of the constant link lengths in equation 4.7d:

$$K_1 = \frac{d}{a} \qquad K_2 = \frac{d}{c} \qquad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac} \tag{4.8a}$$

and:

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4 \tag{4.8b}$$

4

If we substitute the identity $\cos(\theta_2 - \theta_4) = \cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4$, we get the form known as Freudenstein's equation.

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4) \tag{4.8c}$$

In order to reduce equation 4.8b to a more tractable form for solution, it will be useful to substitute the *half-angle identities* which will convert the sin θ_4 and cos θ_4 terms to tan θ_4 terms:

$$\sin\theta_4 = \frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)}; \qquad \cos\theta_4 = \frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} \tag{4.9}$$

This results in the following simplified form, where the link lengths and known input value (θ_2) terms have been collected as constants A, B, and C.

$$A\tan^2\left(\frac{\theta_4}{2}\right) + B\tan\left(\frac{\theta_4}{2}\right) + C = 0 \tag{4.10a}$$

where:

$$A = \cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3$$
$$B = -2\sin\theta_2$$
$$C = K_1 - (K_2 + 1)\cos\theta_2 + K_3$$

Note that equation 4.10a is quadratic in form, and the solution is:

$$\tan\left(\frac{\theta_4}{2}\right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$
$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right) \tag{4.10b}$$

Equation 4.10b has two solutions, obtained from the ± conditions on the radical. These two solutions, as with any quadratic equation, may be of three types: *real and equal*, *real and unequal*, *complex conjugate*. If the discriminant under the radical is negative, then the solution is complex conjugate, which simply means that the link lengths chosen are not capable of connection for the chosen value of the input angle θ_2. This can occur either when the link lengths are completely incapable of connection in any position or, in a non-Grashof linkage, when the input angle is beyond a toggle limit position. There is then no real solution for that value of input angle θ_2. Excepting this situation, the solution will usually be real and unequal, meaning there are two values of θ_4 corresponding to any one value of θ_2. These are referred to as the **crossed** and **open** configurations of the linkage and also as the two **circuits** of the linkage.* In the fourbar linkage, the minus solution gives θ_4 for the open configuration and the positive solution gives θ_4 for the crossed configuration.

Figure 4-5 (p. 182) shows both crossed and open solutions for a Grashof crank-rocker linkage. The terms crossed and open are based on the assumption that the input link 2, for which θ_2 is defined, is placed in the first quadrant (i.e., $0 < \theta_2 < \pi/2$). A Grashof linkage is then defined as **crossed** if the two links adjacent to the shortest link cross one another, and as **open** if they do not cross one another in this position. Note that the configuration of the linkage, either crossed or open, is solely dependent upon the way that the links are assembled. You cannot predict, based on link lengths alone, which of the solutions will be the desired one. In other words, you can obtain either solution with the same linkage by simply taking apart the pin which connects links 3 and 4 in Figure 4-5 (p. 182), and moving those links to the only other positions at which the pin will again connect them. In so doing, you will have switched from one position solution, or **circuit**, to the other.

The solution for angle θ_3 is essentially similar to that for θ_4. Returning to equations 4.6, we can rearrange them to isolate θ_4 on the left side.

$$c\cos\theta_4 = a\cos\theta_2 + b\cos\theta_3 - d \tag{4.6e}$$
$$c\sin\theta_4 = a\sin\theta_2 + b\sin\theta_3 \tag{4.6f}$$

Squaring and adding these equations will eliminate θ_4. The resulting equation can be solved for θ_3 as was done above for θ_4, yielding this expression:

$$K_1\cos\theta_3 + K_4\cos\theta_2 + K_5 = \cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3 \tag{4.11a}$$

The constant K_1 is the same as defined in equation 4.8b, and K_4 and K_5 are:

$$K_4 = \frac{d}{b} \qquad K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab} \tag{4.11b}$$

This also reduces to a quadratic form:

$$D\tan^2\left(\frac{\theta_3}{2}\right) + E\tan\left(\frac{\theta_3}{2}\right) + F = 0 \tag{4.12}$$

where

$$D = \cos\theta_2 - K_1 + K_4\cos\theta_2 + K_5$$
$$E = -2\sin\theta_2$$
$$F = K_1 + (K_4 - 1)\cos\theta_2 + K_5$$

and the solution is:

* See Section 4-13 (p. 208) for a more complete discussion of circuits and branches in linkages.

$$\theta_{3_{1,2}} = 2\arctan\left(\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right) \tag{4.13}$$

As with the angle θ_4, this also has two solutions, corresponding to the crossed and open circuits of the linkage, as shown in Figure 4-5 (p. 182).

EXAMPLE 4-1

Position Analysis of a Fourbar Linkage with the Vector Loop Method.

Problem: Given a fourbar linkage with the link lengths $L_1 = d = 100$ mm, $L_2 = a = 40$ mm, $L_3 = b = 120$ mm, $L_4 = c = 80$ mm. For $\theta_2 = 40°$ find all possible values of θ_3 and θ_4.

Solution: See Figure 4-6 (p. 184) for nomenclature.

1 Using equation 4.8a, calculate the link ratios K_1, K_2 and K_3.

$$\begin{aligned} K_1 &= \frac{d}{a} = \frac{100}{40} = 2.5 \\ K_2 &= \frac{d}{c} = \frac{100}{80} = 1.25 \\ K_3 &= \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \frac{40^2 - 120^2 + 80^2 + 100^2}{2(40)(80)} = 0.562 \end{aligned} \tag{a}$$

2 Use these link ratios to find the intermediate parameters A, B, and C from equation 4.10a.

$$\begin{aligned} A &= \cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3 = \cos(40°) - 2.5 - 1.25\cos(40°) + 0.562 = -2.129 \\ B &= -2\sin\theta_2 = -2\sin(40°) = -1.286 \\ C &= K_1 - (K_2 + 1)\cos\theta_2 + K_3 = 2.5 - (1.25 + 1)\cos(40°) + 0.562 = 1.339 \end{aligned} \tag{b}$$

3 Use equation 4.10b to find θ_4 for both the open and crossed configurations.

$$\begin{aligned} \theta_{4_{open}} &= 2\arctan\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = 2\arctan\left(\frac{1.286 - \sqrt{-1.286^2 - 4(-2.129)(1.339)}}{2(-2.129)}\right) \\ &= 57.33° \\ \theta_{4_{crossed}} &= 2\arctan\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = 2\arctan\left(\frac{1.286 + \sqrt{-1.286^2 - 4(-2.129)(1.339)}}{2(-2.129)}\right) \\ &= -98.01° \end{aligned} \tag{c}$$

4 Use equation 4.11b to find the ratios K_4 and K_5.

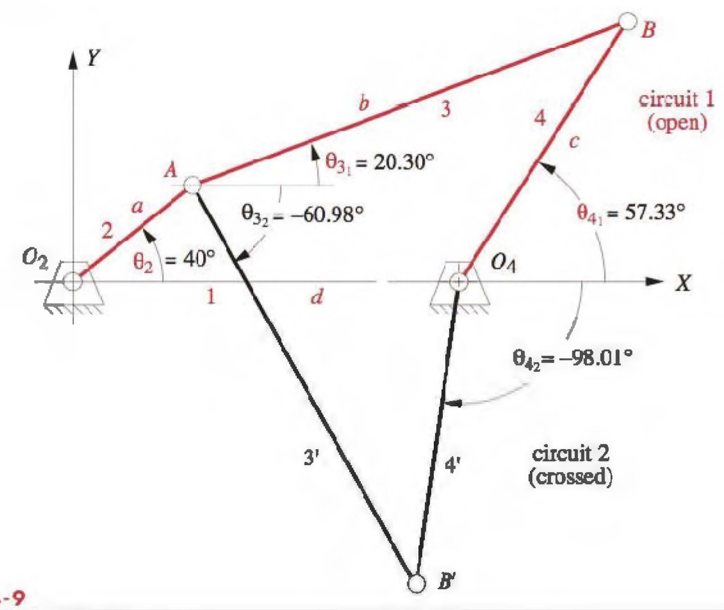


FIGURE 4-9
Solution to Example 4-1

$$K_4 = \frac{d}{b} = \frac{100}{120} = 0.833$$

$$K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab} = \frac{80^2 - 100^2 - 40^2 - 120^2}{2(40)(120)} = -2.042 \qquad (d)$$

5 Use equation 4.12 to find the intermediate parameters D, E, and F.

$$D = \cos\theta_2 - K_1 + K_4\cos\theta_2 + K_5 = \cos(40°) - 2.5 + 0.833(40°) - 2.042 = -3.137$$

$$E = -2\sin\theta_2 = -2\sin(40°) = -1.286 \qquad (e)$$

$$F = K_1 + (K_4 - 1)\cos\theta_2 + K_5 = 2.5 + (0.833 - 1)\cos(40°) - 2.042 = 0.331$$

6 Use equation 4.13 to find θ_3 for both the open and crossed configurations.

$$\theta_{3_{open}} = 2\arctan\left(\frac{-E - \sqrt{E^2 - 4DF}}{2D}\right) = 2\arctan\left(\frac{1.286 - \sqrt{-1.286^2 - 4(-3.137)(0.331)}}{2(-3.137)}\right)$$

$$= 20.30° \qquad (f)$$

$$\theta_{3_{crossed}} = 2\arctan\left(\frac{-E + \sqrt{E^2 - 4DF}}{2D}\right) = 2\arctan\left(\frac{1.286 + \sqrt{-1.286^2 - 4(-3.137)(0.331)}}{2(-3.137)}\right)$$

$$= -60.98°$$

7 The solution is shown in Figure 4-9.

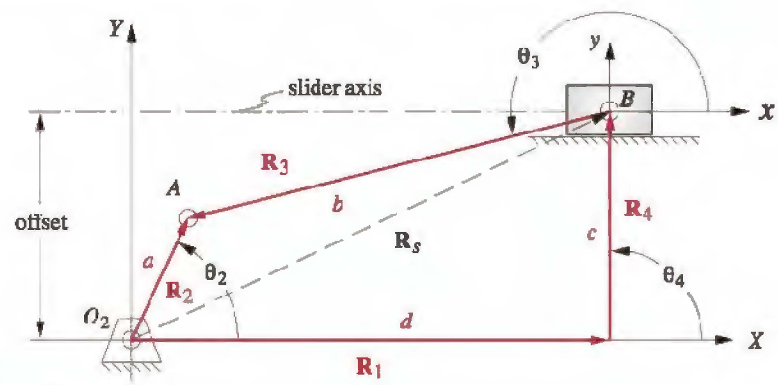


FIGURE 4-10

Position vector loop for a fourbar crank-slider or slider-crank linkage

4.6 THE FOURBAR CRANK-SLIDER POSITION SOLUTION

The same vector loop approach as used for the pure pin-jointed fourbar can be applied to a linkage containing sliders. Figure 4-10 shows an offset fourbar crank-slider linkage, inversion #1. The term **offset** means that *the slider axis extended does not pass through the crank pivot*. This is the general case. (The nonoffset crank-slider linkages shown in Figure 2-13 (p. 52) are the special cases.) This linkage could be represented by only three position vectors, $\mathbf{R}_2$, $\mathbf{R}_3$, and $\mathbf{R}_s$, but one of them ($\mathbf{R}_s$) will be a vector of varying magnitude and angle. It will be easier to use four vectors, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, and $\mathbf{R}_4$ with $\mathbf{R}_1$ arranged parallel to the axis of sliding and $\mathbf{R}_4$ perpendicular. In effect the pair of vectors $\mathbf{R}_1$ and $\mathbf{R}_4$ are orthogonal components of the position vector $\mathbf{R}_s$ from the origin to the slider.

It simplifies the analysis to arrange one coordinate axis parallel to the axis of sliding. The variable-length, constant-direction vector $\mathbf{R}_1$ then represents the slider position with magnitude *d*. The vector $\mathbf{R}_4$ is orthogonal to $\mathbf{R}_1$ and defines the constant magnitude **offset** of the linkage. Note that for the special-case, nonoffset version, the vector $\mathbf{R}_4$ will be zero and $\mathbf{R}_1 = \mathbf{R}_s$. The vectors $\mathbf{R}_2$ and $\mathbf{R}_3$ complete the vector loop. The coupler's position vector $\mathbf{R}_3$ is placed with its root at the slider which then defines its angle θ_3 at point *B*. This particular arrangement of position vectors leads to a vector loop equation similar to the pin-jointed fourbar example:

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.14a}$$

Compare equation 4.14a to equation 4.5a (p. 186) and note that the only difference is the sign of $\mathbf{R}_3$. This is due solely to the somewhat arbitrary choice of the sense of the position vector $\mathbf{R}_3$ in each case. The angle θ_3 must always be measured at the root of vector $\mathbf{R}_3$, and in this example it will be convenient to have that angle θ_3 at the joint labeled *B*. Once these arbitrary choices are made it is crucial that the resulting algebraic signs be carefully observed in the equations, or the results will be completely erroneous. Letting the vector magnitudes (link lengths) be represented by *a, b, c, d* as shown, we can substitute the complex number equivalents for the position vectors.

$$ae^{j\theta_2} - be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0 \tag{4.14b}$$

Substitute the Euler equivalents:

$$a(\cos\theta_2 + j\sin\theta_2) - b(\cos\theta_3 + j\sin\theta_3)$$
$$-c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0 \tag{4.14c}$$

Separate the real and imaginary components:

real part (x component):

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d\cos\theta_1 = 0$$

but: $\theta_1 = 0$, so: $$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0 \tag{4.15a}$$

imaginary part (y component):

$$ja\sin\theta_2 - jb\sin\theta_3 - jc\sin\theta_4 - jd\sin\theta_1 = 0$$

but: $\theta_1 = 0$, and the j's divide out, so: (4.15b)

$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0$$

We want to solve equations 4.15 simultaneously for the two unknowns, link length d and link angle θ_3. The independent variable is crank angle θ_2. Link lengths a and b, the offset c, and angle θ_4 are known. But note that since we set up the coordinate system to be parallel and perpendicular to the axis of the slider block, the angle θ_1 is zero and θ_4 is 90°. Equation 4.15b can be solved for θ_3 and the result substituted into equation 4.15a to solve for d. The solution is:

$$\theta_{3_1} = \arcsin\left(\frac{a\sin\theta_2 - c}{b}\right) \tag{4.16a}$$

$$d = a\cos\theta_2 - b\cos\theta_3 \tag{4.16b}$$

Note that there are again two valid solutions corresponding to the two circuits of the linkage. The arcsine function is multivalued. Its evaluation will give a value between ±90° representing only one circuit of the linkage. The value of d is dependent on the calculated value of θ_3. The value of θ_3 for the second circuit of the linkage can be found from:

$$\theta_{3_2} = \arcsin\left(-\frac{a\sin\theta_2 - c}{b}\right) + \pi \tag{4.17}$$

EXAMPLE 4-2

Position Analysis of a Fourbar Crank-Slider Linkage with the Vector Loop Method.

Problem: Given a fourbar crank-slider linkage with the link lengths $L_2 = a = 40$ mm, $L_3 = b = 120$ mm, *offset* $= c = -20$ mm. For $\theta_2 = 60°$ find all possible values of θ_3 and slider position d.

Solution: (See Figure 4-10 for nomenclature.)

1 Using equation 4.16a, calculate the link coupler angle θ_3 for the open configuration.

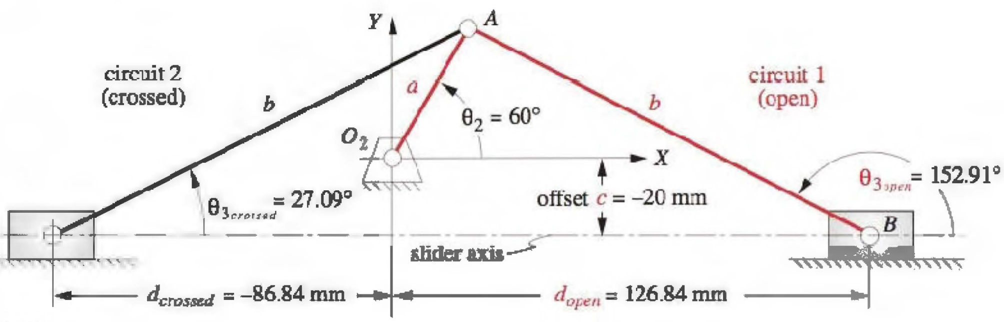


FIGURE 4-11
Solution to Example 4-2

$$\theta_{3_{open}} = \arcsin\left(\frac{a\sin\theta_2 - c}{b}\right) = \arcsin\left(\frac{40\sin(60°) - (-20)}{120}\right) = 152.91° \qquad (a)$$

2 Using equation 4.16b and the result from step 1, calculate slider position d for open linkage.

$$d = a\cos\theta_2 - b\cos\theta_3 = 40\cos(60°) - 120\cos(152.91°) = 126.84 \text{ mm} \qquad (b)$$

3 Using equation 4.17, calculate the link coupler angle θ_3 for the crossed configuration.

$$\theta_{3_{crossed}} = \arcsin\left(-\frac{a\sin\theta_2 - c}{b}\right) + \pi = \arcsin\left(-\frac{40\sin(60°) - (-20)}{120}\right) + \pi = 27.09° \qquad (c)$$

4 Using equation 4.16b and the result from step 3, calculate slider position d for crossed linkage.

$$d = a\cos\theta_2 - b\cos\theta_3 = 40\cos(60°) - 120\cos(27.09°) = -86.84 \text{ mm} \qquad (d)$$

5 Note that θ_3 is measured at the slider end of the coupler as shown in Figure 4-11.

4.7 THE FOURBAR SLIDER-CRANK POSITION SOLUTION

The *fourbar* ***slider-crank*** *linkage* has the same geometry as the *fourbar* ***crank-slider*** *linkage* that was analyzed in the previous section. The name change indicates that it will be driven with the slider as input and the crank as output. This is sometimes referred to as a "back-driven" crank-slider. We will use the term **slider-crank** to define it as slider-driven. This is a very commonly used linkage configuration. Every internal-combustion piston engine has as many of these as it has cylinders. The vector loop is as shown in Figure 4-10 (p. 192) and the vector loop equation is identical to equation 4.14a. But now we must solve this equation for θ_2 as a function of slider position d.

Start with equation 4.14a, make the substitutions of equation 4.14b and the simplifications of equations 4.15 to get the same simultaneous equation set:

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0 \tag{4.15a}$$
$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0 \tag{4.15b}$$

but

$$\theta_4 = 90° \quad \therefore \quad \sin\theta_4 = 1, \ \cos\theta_4 = 0$$

so

$$a\cos\theta_2 - b\cos\theta_3 - d = 0 \tag{4.18a}$$
$$a\sin\theta_2 - b\sin\theta_3 - c = 0 \tag{4.18b}$$

As was done in the fourbar linkage solution, isolate the θ_3 terms on one side, square both equations, and add them to eliminate θ_3.

$$b\cos\theta_3 = a\cos\theta_2 - d$$
$$b\sin\theta_3 = a\sin\theta_2 - c$$

square:
$$b^2\cos^2\theta_3 = (a\cos\theta_2 - d)^2$$
$$b^2\sin^2\theta_3 = (a\sin\theta_2 - c)^2$$

add:
$$b^2\left(\sin^2\theta_3 + \cos^2\theta_3\right) = (a\cos\theta_2 - d)^2 + (a\sin\theta_2 - c)^2$$
$$b^2 = (a\cos\theta_2 - d)^2 + (a\sin\theta_2 - c)^2$$
$$b^2 = a^2\cos^2\theta_2 - 2ad\cos\theta_2 + d^2 + a^2\sin^2\theta_2 - 2ac\sin\theta_2 + c^2$$
$$b^2 = a^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2ad\cos\theta_2 - 2ac\sin\theta_2 + c^2 + d^2$$
$$a^2 - b^2 + c^2 + d^2 - 2ac\sin\theta_2 - 2ad\cos\theta_2 = 0 \tag{4.19}$$

To simplify, create some constant parameters:

let
$$K_1 = a^2 - b^2 + c^2 + d^2, \ K_2 = -2ac, \ K_3 = -2ad$$

then
$$K_1 + K_2\sin\theta_2 + K_3\cos\theta_2 = 0 \tag{4.20}$$

As we did for the fourbar linkage, substitute the tangent half-angle identities (equation 4.9) for $\sin\theta_2$ and $\cos\theta_2$ to get the equation in terms of one trigonometric function.

$$K_1 + K_2\left(\frac{2\tan\dfrac{\theta_2}{2}}{1+\tan^2\dfrac{\theta_2}{2}}\right) + K_3\left(\frac{1-\tan^2\dfrac{\theta_2}{2}}{1+\tan^2\dfrac{\theta_2}{2}}\right) = 0$$

simplify
$$(K_1 - K_3)\tan^2\frac{\theta_2}{2} + 2K_2\tan\frac{\theta_2}{2} + (K_1 + K_3) = 0$$

let
$$A = K_1 - K_3, \ B = 2K_2, \ C = K_1 + K_3$$

then
$$A\tan^2\frac{\theta_2}{2} + B\tan\frac{\theta_2}{2} + C = 0$$

and
$$\theta_{2_{1,2}} = 2\arctan\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right) \tag{4.21}$$

Once θ_2 is known for a given value of d, θ_3 can be found from either equation 4.18a or 4.18b.

Note that there are two solutions to equation 4.21 representing the two branches of the linkage on the circuit to which the given value of slider position d applies.* The equation will fail when the backdriven slider-crank is at either top dead center (TDC) or bottom dead center (BDC). These are indeterminate change points between the branches at which the mathematics cannot predict which branch the linkage will go to next. A real slider-crank linkage can only make a full revolution of the crank if there is some stored energy in the crank to carry it through the dead centers twice per revolution. This is why you must spin a piston engine to start it and why they typically have a flywheel attached to the crankshaft to provide the angular momentum needed to pass through TDC and BDC.

* The crank-slider and slider-crank linkage both have two circuits or configurations in which they can be independently assembled, sometimes called open and crossed. Because effective link 4 is always perpendicular to the slider axis, it is parallel to itself on both circuits. This results in the two circuits being mirror images of one another, mirrored about a line through the crank pivot and perpendicular to the slide axis. Thus, the choice of value of slider position d in the calculation of the slider-crank linkage determines which circuit is being analyzed. But, because of the change points at TDC and BDC, the slider-crank has two branches on each circuit, and the two solutions obtained from equation 4.21 represent the two branches on the one circuit being analyzed. In contrast, the crank-slider has only one branch per circuit because when the crank is driven, it can make a full revolution and there are no change points to separate branches. See Section 4.13 (p. 208) for a more complete discussion of circuits and branches in linkages.

EXAMPLE 4-3

Position Analysis of a Fourbar Slider-Crank Linkage with the Vector Loop Method

Problem: Given a fourbar slider-crank linkage with the link lengths $L_2 = a = 40$ mm, $L_3 = b = 120$ mm, *offset* $= c = -20$ mm. For $d = 100$ mm, find all possible values of θ_2 and θ_3 on the circuit defined by the given value of d.

Solution: See Figure 4-9 (p. 191) for nomenclature.

1 Find the TDC and BDC positions of the linkage.

$$\begin{aligned} d_{BDC} &= b - a = 120 - 40 = 80 \text{ mm} \\ d_{TDC} &= b + a = 120 + 40 = 160 \text{ mm} \end{aligned} \qquad (a)$$

The requested position of $d = 100$ mm is within the range of motion of the slider-crank linkage and is neither TDC nor BDC, so equations 4.20 and 4.21 can be used.

2 Find the intermediate parameters needed from equations 4.20 and 4.21.

$$\begin{aligned} K_1 &= a^2 - b^2 + c^2 + d^2 = 40^2 - 120^2 + (-20)^2 + 100^2 = -2400 \\ K_2 &= -2ac = -2(40)(-20) = 1600 \\ K_3 &= -2ad = -2(40)(100) = -8000 \\ A &= K_1 - K_3 = -2400 - (-8000) = 5600 \\ B &= 2K_2 = 2(1600) = 3200 \\ C &= K_1 + K_3 = -2400 + (-8000) = -10400 \end{aligned} \qquad (b)$$

3 Find the two values of θ_2 from equation 4.21.

$$\begin{aligned} \theta_{2_1} &= 2\tan^{-1}\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = 2\tan^{-1}\left(\frac{-3200 + \sqrt{3200^2 - 4(5600)(-10400)}}{2(5600)}\right) = 95.798° \\ \theta_{2_2} &= 2\tan^{-1}\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = 2\tan^{-1}\left(\frac{-3200 - \sqrt{3200^2 - 4(5600)(-10400)}}{2(5600)}\right) = -118.418° \end{aligned} \qquad (c)$$

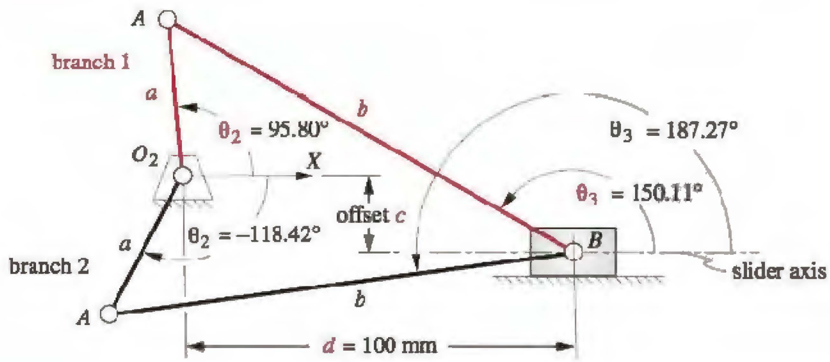


FIGURE 4-12
Solution to Example 4-3

4 Find the two values of θ_3 from either equation 4.16a or 4.17. Calculate θ_3 with both equations for one value of θ_2 and then use equation 4.16b with that result to determine which of the two equations gives the correct value of d to match the circuit of this linkage. Then use that equation with each of the θ_2 values to get the correct values of θ_3 for each branch of this circuit. This example needs equation 4.17 for its circuit.

$$\theta_{3_1} = \sin^{-1}\left(-\frac{a\sin\theta_{2_1} - c}{b}\right) + \pi = \sin^{-1}\left(-\frac{40\sin(95.798°) - (-20)}{120}\right) + \pi = 150.113°$$
$$\theta_{3_2} = \cos^{-1}\left(\frac{a\sin\theta_{2_2} - c}{b}\right) + \pi = \cos^{-1}\left(\frac{40\sin(-118.418°) - (-20)}{120}\right) + \pi = 187.267° \qquad (d)$$

5 The solution is shown in Figure 4-12.

4.8 AN INVERTED CRANK-SLIDER POSITION SOLUTION

Figure 4-13a* (p. 198) shows inversion #3 of the common fourbar crank-slider linkage in which the sliding joint is between links 3 and 4 at point B. This is shown as an **offset** crank-slider mechanism. The slider block has pure rotation with its center offset from the slide axis. (Figure 2-13c, p. 52, shows the nonoffset version of this linkage in which the vector $\mathbf{R}_4$ is zero.)

The global coordinate system is again taken with its origin at input crank pivot O_2 and the positive X axis along link 1, the ground link. A local axis system has been placed at point B in order to define θ_3. Note that there is a fixed angle γ within link 4 which defines the slot angle with respect to that link.

In Figure 4-13b (p. 198), the links have been represented as position vectors having senses consistent with the coordinate systems that were chosen for convenience in defining the link angles. This particular arrangement of position vectors leads to the same vector loop equation as the previous crank-slider example.

* This figure is provided as animated AVI and Working Model files on the DVD. Its filename is the same as the figure number.

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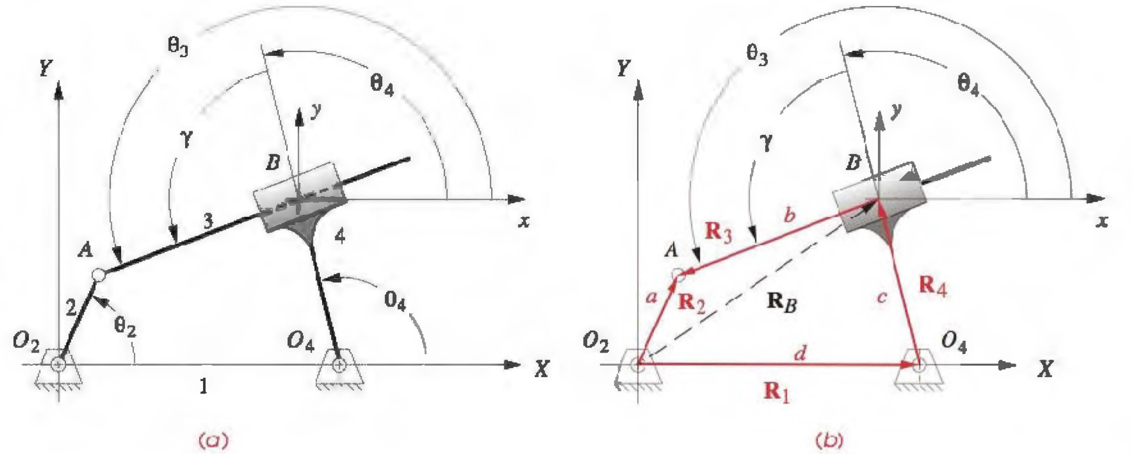


FIGURE 4-13
Inversion #3 of the slider-crank fourbar linkage

Equations 4.14 and 4.15 (pp. 192–193) apply to this inversion as well. Note that the absolute position of point B is defined by vector $\mathbf{R}_B$ which varies in both magnitude and direction as the linkage moves. We choose to represent $\mathbf{R}_B$ as the vector difference $\mathbf{R}_2 - \mathbf{R}_3$ in order to use the actual links as the position vectors in the loop equation.

All slider linkages will have at least one link whose effective length between joints will vary as the linkage moves. In this example the length of link 3 between points A and B, designated as b, will change as it passes through the slider block on link 4. Thus the value of b will be one of the variables to be solved for in this inversion. Another variable will be θ_4, the angle of link 4. Note however, that we also have an unknown in θ_3, the angle of link 3. This is a total of three unknowns. Equations 4.15 can only be solved for two unknowns. Thus we require another equation to solve the system. There is a fixed relationship between angles θ_3 and θ_4, shown as γ in Figure 4-10 (p. 192), which gives the equation:

$$\theta_3 = \theta_4 + \gamma \tag{4.22}$$

Repeating equations 4.15 and renumbering them for the reader's convenience:

$$a\cos\theta_2 - b\cos\theta_3 - c\cos\theta_4 - d = 0 \tag{4.23a}$$

$$a\sin\theta_2 - b\sin\theta_3 - c\sin\theta_4 = 0 \tag{4.23b}$$

These have only two unknowns and can be solved simultaneously for θ_4 and b. Equation 4.23b can be solved for link length b and substituted into equation 4.23a.

$$b = \frac{a\sin\theta_2 - c\sin\theta_4}{\sin\theta_3} \tag{4.24a}$$

$$a\cos\theta_2 - \frac{a\sin\theta_2 - c\sin\theta_4}{\sin\theta_3}\cos\theta_3 - c\cos\theta_4 - d = 0 \tag{4.24b}$$

Substitute equation 4.22 and after some algebraic manipulation, equation 4.24 can be reduced to:

$$P\sin\theta_4 + Q\cos\theta_4 + R = 0$$

where (4.25)

$$P = a\sin\theta_2\sin\gamma + (a\cos\theta_2 - d)\cos\gamma$$
$$Q = -a\sin\theta_2\cos\gamma + (a\cos\theta_2 - d)\sin\gamma$$
$$R = -c\sin\gamma$$

Note that the factors P, Q, R are constant for any input value of θ_2. To solve this for θ_4, it is convenient to substitute the tangent half angle identities (equation 4.9, p. 188) for the sin θ_4 and cos θ_4 terms. This will result in a quadratic equation in tan $(\theta_4 / 2)$ which can be solved for the two values of θ_4.

$$P\frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} + Q\frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} + R = 0 \tag{4.26a}$$

This reduces to:

$$(R-Q)\tan^2\left(\frac{\theta_4}{2}\right) + 2P\tan\left(\frac{\theta_4}{2}\right) + (Q+R) = 0$$

let

$$S = R - Q, \qquad T = 2P, \qquad U = Q + R$$

then

$$S\tan^2\left(\frac{\theta_4}{2}\right) + T\tan\left(\frac{\theta_4}{2}\right) + U = 0 \tag{4.26b}$$

and the solution is:

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-T \pm \sqrt{T^2 - 4SU}}{2S}\right) \tag{4.26c}$$

As was the case with the previous examples, this also has a crossed and an open solution represented by the plus and minus signs on the radical. Note that we must also calculate the values of link length b for each θ_4 by using equation 4.24a. The coupler angle θ_3 is found from equation 4.22 (p. 197).

4.9 LINKAGES OF MORE THAN FOUR BARS

With some exceptions,* the same approach as shown here for the fourbar linkage can be used for any number of links in a closed-loop configuration. More complicated linkages may have multiple loops which will lead to more equations to be solved simultaneously and may require an iterative solution. Alternatively, Wampler [10] presents a new, general, noniterative method for the analysis of planar mechanisms containing any number of rigid links connected by rotational and/or translational joints.

4

* Waldron and Sreenivasan[1] report that the common solution methods for position analysis are not general, i.e., are not extendable to *n*-link mechanisms. Conventional position analysis methods, such as those used here, rely on the presence of a fourbar loop in the mechanism that can be solved first, followed by a decomposition of the remaining links into a series of dyads. Not all mechanisms contain fourbar loops. (One eightbar, 1-*DOF* linkage contains no fourbar loops—see the 16th isomer at lower right in Figure 2-11d on p. 50). Even if there is a fourbar loop, its pivots may not be grounded, requiring that the linkage be inverted to start the solution. Also, if the driving joint is not in the fourbar loop, then interpolation is needed to solve for link positions.

The Geared Fivebar Linkage

Another example, which can be reduced to two equations in two unknowns, is the **geared fivebar linkage**, which was introduced in Section 2.14 (p. 62) and is shown in Figure 4-14a and program LINKAGES disk file F04-11.5br. The vector loop for this linkage is shown in Figure 4-14b. It obviously has one more position vector than the fourbar. Its vector loop equation is:

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_5 - \mathbf{R}_1 = 0 \tag{4.27a}$$

4

Note that the vector senses are again chosen to suit the analyst's desires to have the vector angles defined at a convenient end of the respective link. Equation 4.27b substitutes the complex polar notation for the position vectors in equation 4-23a, using *a*, *b*, *c*, *d*, *f* to represent the scalar lengths of the links as shown in Figure 4-14.

$$a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_5} - f e^{j\theta_1} = 0 \tag{4.27b}$$

Note also that this vector loop equation has three unknown variables in it, namely the angles of links 3, 4, and 5. (The angle of link 2 is the input, or independent, variable, and link 1 is fixed with constant angle.) Since a two-dimensional vector equation can only be solved for two unknowns, we will need another equation to solve this system. Because this is a geared fivebar linkage, there exists a relationship between the two geared links, here links 2 and 5. Two factors determine how link 5 behaves with respect to link 2, namely, the **gear ratio** λ and the **phase angle** ϕ. The relationship is:

$$\theta_5 = \lambda\theta_2 + \phi \tag{4.27c}$$

This allows us to express θ_5 in terms of θ_2 in equation 4.27b and reduce the unknowns to two by substituting equation 4.27c into equation 4.27b.

$$a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j(\lambda\theta_2 + \phi)} - f e^{j\theta_1} = 0 \tag{4.28a}$$

Note that the gear ratio λ is the ratio of the diameters of the gears connecting the two links ($\lambda = dia_2 \,/\, dia_5$), and the phase angle ϕ is the *initial angle* of link 5 with respect to link 2. When link 2 is at zero degrees, link 5 is at the **phase angle** ϕ. Equation 4.27c defines the relationship between θ_2 and θ_5. Both λ and ϕ are design parameters selected by the design engineer along with the link lengths. With these parameters defined, the only unknowns left in equation 4.28 are θ_3 and θ_4.

The behavior of the geared fivebar linkage can be modified by changing the link lengths, the gear ratio, or the phase angle. The phase angle can be changed simply by lifting the gears out of engagement, rotating one gear with respect to the other, and re-engaging them. Since links 2 and 5 are rigidly attached to gears 2 and 5, respectively, their relative angular rotations will be changed also. It is this fact that results in different positions of links 3 and 4 with any change in phase angle. The coupler curve's shapes will also change with variation in any of these parameters as can be seen in Figure 3-23 (p. 131) and in Appendix E.

The procedure for solution of this vector loop equation is the same as that used for the fourbar linkage:

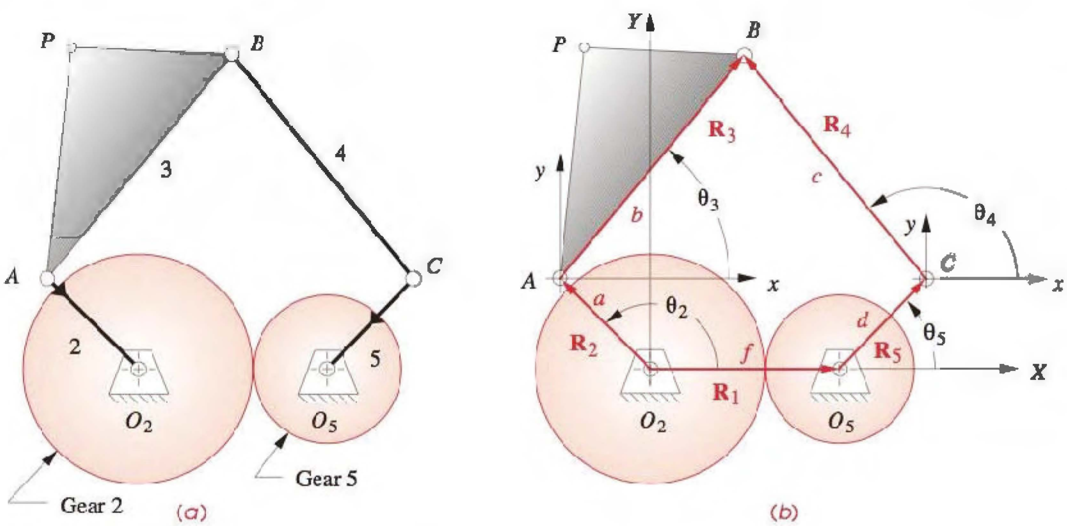


FIGURE 4-14

The geared fivebar linkage and its vector loop

1 Substitute the Euler equivalent (equation 4.4a, p. 185) into each term in the vector loop equation 4.28a.

$$\begin{aligned}&a(\cos\theta_2 + j\sin\theta_2)+b(\cos\theta_3 + j\sin\theta_3)-c(\cos\theta_4 + j\sin\theta_4)\\&-d\left[\cos(\lambda\theta_2+\phi)+j\sin(\lambda\theta_2+\phi)\right]-f(\cos\theta_1 + j\sin\theta_1)=0\end{aligned} \tag{4.28b}$$

2 Separate the real and imaginary parts of the cartesian form of the vector loop equation.

$$a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d\cos(\lambda\theta_2+\phi) - f\cos\theta_1 = 0 \tag{4.28c}$$

$$a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 - d\sin(\lambda\theta_2+\phi) - f\sin\theta_1 = 0 \tag{4.28d}$$

3 Rearrange to isolate one unknown (either θ_3 or θ_4) in each scalar equation. Note that θ_1 is zero.

$$b\cos\theta_3 = -a\cos\theta_2 + c\cos\theta_4 + d\cos(\lambda\theta_2+\phi) + f \tag{4.28e}$$

$$b\sin\theta_3 = -a\sin\theta_2 + c\sin\theta_4 + d\sin(\lambda\theta_2+\phi) \tag{4.28f}$$

4 Square both equations and add them to eliminate one unknown, say θ_3.

$$\begin{aligned}b^2 = {}&2c\left[d\cos(\lambda\theta_2+\phi) - a\cos\theta_2 + f\right]\cos\theta_4\\&+2c\left[d\sin(\lambda\theta_2+\phi) - a\sin\theta_2\right]\sin\theta_4\\&+a^2+c^2+d^2+f^2-2af\cos\theta_2\\&-2d(a\cos\theta_2 - f)\cos(\lambda\theta_2+\phi)\\&-2ad\sin\theta_2\sin(\lambda\theta_2+\phi)\end{aligned} \tag{4.28g}$$

5 Substitute the tangent half-angle identities (equation 4.9, p. 188) for the sine and cosine terms and manipulate the resulting equation in the same way as was done for the fourbar linkage in order to solve for θ_4.

$$\begin{aligned}
A &= 2c\left[d\cos(\lambda\theta_2+\phi)-a\cos\theta_2+f\right]\\
B &= 2c\left[d\sin(\lambda\theta_2+\phi)-a\sin\theta_2\right]\\
C &= a^2-b^2+c^2+d^2+f^2-2af\cos\theta_2\\
&\quad -2d(a\cos\theta_2-f)\cos(\lambda\theta_2+\phi)-2ad\sin\theta_2\sin(\lambda\theta_2+\phi)\\
D &= C-A, \qquad E=2B, \qquad F=A+C
\end{aligned}$$

$$\theta_{4_{1,2}} = 2\arctan\left(\frac{-E\pm\sqrt{E^2-4DF}}{2D}\right) \tag{4.28h}$$

6 Repeat steps 3 to 5 for the other unknown angle θ_3.

$$\begin{aligned}
G &= 2b\left[a\cos\theta_2-d\cos(\lambda\theta_2+\phi)-f\right]\\
H &= 2b\left[a\sin\theta_2-d\sin(\lambda\theta_2+o)\right]\\
K &= a^2+b^2-c^2+d^2+f^2-2af\cos\theta_2\\
&\quad -2d(a\cos\theta_2-f)\cos(\lambda\theta_2+\phi)\\
&\quad -2ad\sin\theta_2\sin(\lambda\theta_2+\phi)\\
L &= K-G; \quad M=2H; \quad N=G+K
\end{aligned}$$

$$\theta_{3_{1,2}} = 2\arctan\left(\frac{-M\pm\sqrt{M^2-4LN}}{2L}\right) \tag{4.28i}$$

Note that these derivation steps are essentially identical to those for the pin-jointed fourbar linkage once θ_2 is substituted for θ_5 using equation 4.27c (p. 200).

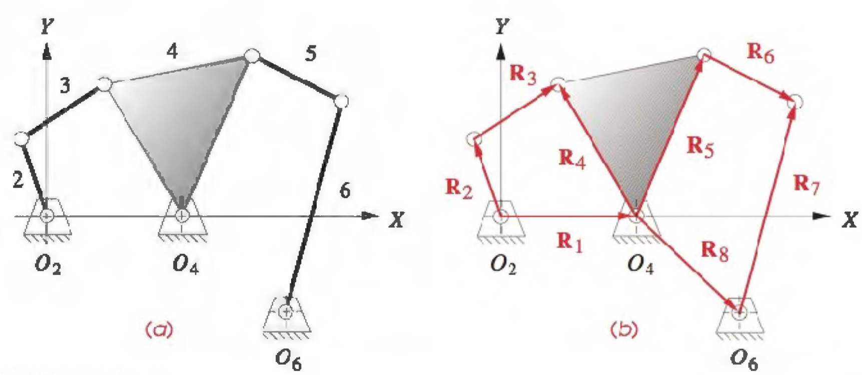


FIGURE 4-15

Watt's sixbar linkage and vector loop

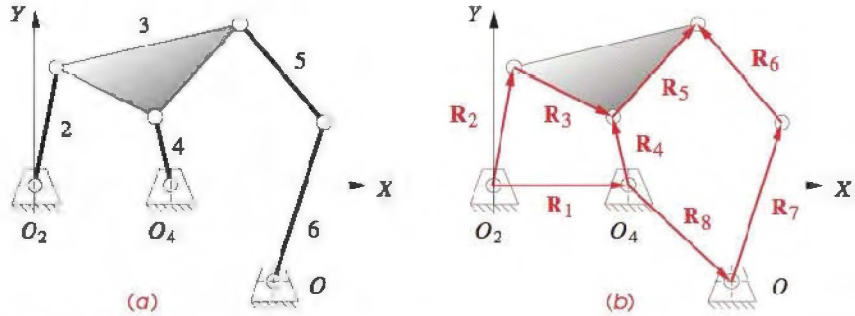


FIGURE 4-16

Stephenson's sixbar linkage and vector loops

Sixbar Linkages

WATT'S SIXBAR is essentially two fourbar linkages in series, as shown in Figure 4-15a, and can be analyzed as such. Two vector loops are drawn as shown in Figure 4-15b. These vector loop equations can be solved in succession with the results of the first loop applied as input to the second loop. Note that there is a constant angular relationship between vectors $\mathbf{R}_4$ and $\mathbf{R}_5$ within link 4. The solution for the fourbar linkage (equations 4.10 and 4.13, pp. 188 and 189, respectively) is simply applied twice in this case. Depending on the inversion of the Watts linkage being analyzed, there may be two four-link loops or one four-link and one five-link loop. (See Figure 2-14, p. 54.) In either case, if the four-link loop is analyzed first, there will not be more than two unknown link angles to be found at one time.

STEPHENSON'S SIXBAR is a more complicated mechanism to analyze. Two vector loops can be drawn, but depending on the inversion being analyzed, either one or both loops will have five links* and three unknown angles as shown in Figure 4-13a and b (p. 198). However, the two loops will have at least one nonground link in common and so a solution can be found. In the other cases an iterative solution such as a Newton-Raphson method (see Section 4.14, p. 210) must be used to find the roots of the equations. Program LINKAGES is limited to the inversions which allow a closed-form solution, one of which is shown in Figure 4-16, and it does not do the iterative solution.

4.10 POSITION OF ANY POINT ON A LINKAGE

Once the angles of all the links are found, it is simple and straightforward to define and calculate the position of any point on any link for any input position of the linkage. Figure 4-17 shows a fourbar linkage whose coupler, link 3, is enlarged to contain a coupler point P. The crank and rocker have also been enlarged to show points S and U which might represent the centers of gravity of those links. We want to develop algebraic expressions for the positions of these (or any) points on the links.

* See footnote on p. 199.

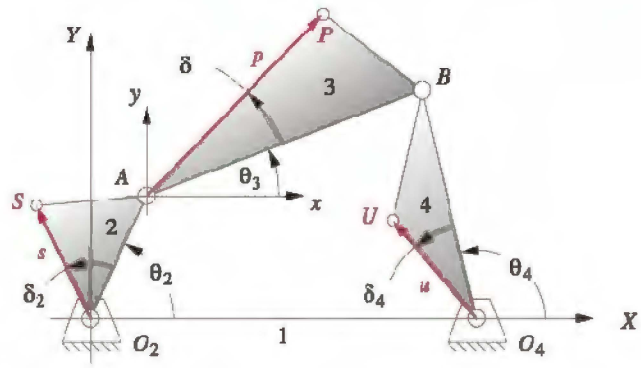


FIGURE 4-17
Positions of points on the links

To find the position of point S, draw a position vector from the fixed pivot O_2 to point S. This vector $\mathbf{R}_{SO_2}$ makes an angle δ_2 with the vector $\mathbf{R}_{AO_2}$. This angle δ_2 is completely defined by the geometry of link 2 and is constant. The position vector for point S is then:

$$\mathbf{R}_{SO_2} = \mathbf{R}_S = se^{j(\theta_2 - \delta_2)} = s\left[\cos(\theta_2 + \delta_2) + j\sin(\theta_2 + \delta_2)\right] \tag{4.29}$$

The position of point U on link 4 is found in the same way, using the angle δ_4 which is a constant angular offset within the link. The expression is:

$$\mathbf{R}_{UO_4} = ue^{j(\theta_4 + \delta_4)} = u\left[\cos(\theta_4 + \delta_4) + j\sin(\theta_4 + \delta_4)\right] \tag{4.30}$$

The position of point P on link 3 can be found from the addition of two position vectors $\mathbf{R}_A$ and $\mathbf{R}_{PA}$. Vector $\mathbf{R}_A$ is already defined from our analysis of the link angles in equation 4.5 (p. 186). Vector $\mathbf{R}_{PA}$ is the relative position of point P with respect to point A. Vector $\mathbf{R}_{PA}$ is defined in the same way as $\mathbf{R}_S$ or $\mathbf{R}_U$, using the internal link offset angle δ_3 and the position angle of link 3, θ_3.

$$\mathbf{R}_{PA} = pe^{j(\theta_3 + \delta_3)} = p\left[\cos(\theta_3 + \delta_3) + j\sin(\theta_3 + \delta_3)\right] \tag{4.31a}$$

$$\mathbf{R}_P = \mathbf{R}_A + \mathbf{R}_{PA} \tag{4.31b}$$

Compare equation 4.31b with equations 4.1 (p. 178). Equation 4.31b is the position difference equation.

4.11 TRANSMISSION ANGLES

The transmission angle was defined in Section 3.3 (p. 100) for a fourbar linkage. That definition is repeated here for your convenience.

The **transmission angle** μ is shown in Figure 3-3a (p. 102) and is defined as *the angle between the output link and the coupler*. It is usually taken as the absolute value of the acute angle of the pair of angles at the intersection of the two links and varies continuously from some minimum to some maximum value as the linkage goes through its range of motion. It is a measure of the quality of force transmission at the joint.*

We will expand that definition here to represent the angle between any two links in a linkage, as a linkage can have many transmission angles. The angle between any output link and the coupler which drives it is a transmission angle. Now that we have developed the analytic expressions for the angles of all the links in a mechanism, it is easy to define the transmission angle algebraically. It is merely the difference between the angles of the two joined links through which we wish to pass some force or velocity. For our fourbar linkage example it will be the difference between θ_3 and θ_4. By convention we take the absolute value of the difference and force it to be an acute angle.

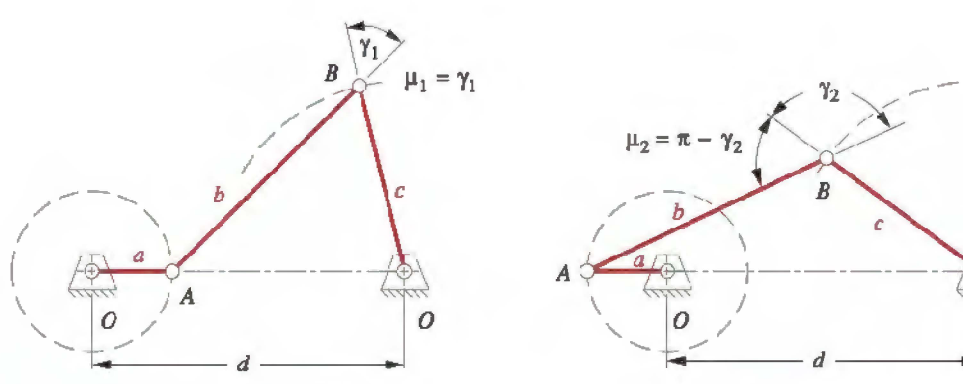
$$\theta_{trans} = |\theta_3 - \theta_4|$$
$$\text{if} \quad \theta_{trans} > \frac{\pi}{2} \qquad \text{then} \quad \mu = \pi - \theta_{trans} \quad \text{else} \quad \mu = \theta_{trans} \tag{4.32}$$

This computation can be done for any joint in a linkage by using the appropriate link angles.

Extreme Values of the Transmission Angle

For a Grashof crank-rocker fourbar linkage the minimum value of the transmission angle will occur when the crank is colinear with the ground link as shown in Figure 4-18. The values of the transmission angle in these positions are easily calculated from the law of cosines since the linkage is then in a triangular configuration. The sides of the two triangles are link 3, link 4, and either the sum or difference of links 1 and 2. Depending on the linkage geometry, the minimum value of the transmission angle μ_{min} will occur either when links 1 and 2 are *colinear and overlapping* as shown in Figure 4-18a or when links 1 and 2 are *colinear and nonoverlapping* as shown in Figure 4-18b. Using notation consistent with Section 4.5 (p. 183) and Figure 4-7 (p. 194) we will label the links:

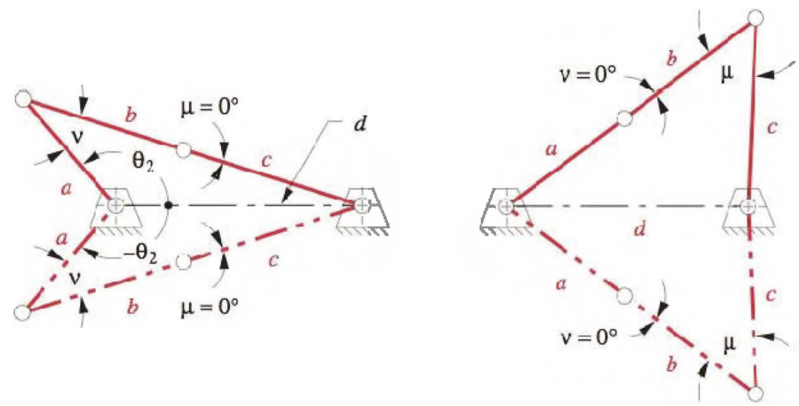
* The transmission angle has limited application. It only predicts the quality of force or torque transmission if the input and output links are pivoted to ground. If the output force is taken from a floating link (coupler), then the transmission angle is of no value. A different index of merit called the joint force index (JFI) is presented in Chapter 11 which discusses force analysis in linkages. (See Section 11.12 p. 611.) The JFI is useful for situations in which the output link is floating as well as giving the same kind of information when the output is taken from a link rotating against the ground. However, the JFI requires a complete force analysis of the linkage be done whereas the transmission angle is determined from linkage geometry alone.



(a) Overlapped (b) Extended

FIGURE 4-18

The minimum transmission angle in the Grashof crank-rocker fourbar linkage occurs in one of two positions



(a) Toggle positions for links b and c

(b) Toggle postions for links a and b

FIGURE 4-19

Non-Grashof triple-rocker linkages in toggle

a = link 2, b = link 3, c = link 4, d = link 1

For the overlapping case (Figure 4-15a, p. 202) the cosine law gives

$$\mu_1 = \gamma_1 = \arccos\left[\frac{b^2 + c^2 - (d-a)^2}{2bc}\right] \tag{4.33a}$$

and for the extended case, the cosine law gives

$$\mu_2 = \pi - \gamma_2 = \pi - \arccos\left[\frac{b^2 + c^2 - (d+a)^2}{2bc}\right] \tag{4.33b}$$

The minimum transmission angle μ_{min} in a Grashof crank-rocker linkage is then the smaller of μ_1 and μ_2.

For a **Grashof double-rocker** linkage the transmission angle can vary from 0 to 90 degrees because the coupler can make a full revolution with respect to the other links. For a **non-Grashof triple-rocker** linkage the transmission angle will be zero degrees in the toggle positions which occur when the output rocker c and the coupler b are colinear as shown in Figure 4-19a. In the other toggle positions when input rocker a and coupler b are colinear (Figure 4-19b), the transmission angle can be calculated from the cosine law as:

when $\quad \nu = 0,$

$$\mu = \arccos\left[\frac{(a+b)^2 + c^2 - d^2}{2c(a+b)}\right] \tag{4.34}$$

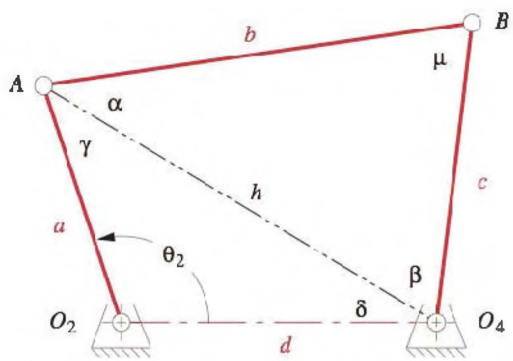


FIGURE 4-20
Finding the crank angle corresponding to the toggle positions

This is not the smallest value that the transmission angle μ can have in a triple-rocker, as that will obviously be zero. Of course, when analyzing any linkage, the transmission angles can easily be computed and plotted for all positions using equation 4.32. Program LINKAGES does this. The student should investigate the variation in transmission angle for the example linkages in those programs. Disk file F04-15.4br can be opened in program LINKAGES to observe that linkage in motion.

4.12 TOGGLE POSITIONS

The input link angles which correspond to the toggle positions (stationary configurations) of the **non-Grashof triple-rocker** can be calculated by the following method, using trigonometry. Figure 4-20 shows a non-Grashof fourbar linkage in a general position. A construction line h has been drawn between points A and O_4. This divides the quadrilateral loop into two triangles, O_2AO_4 and ABO_4. Equation 4.35 uses the cosine law to express the transmission angle μ in terms of link lengths and the input link angle θ_2.

$$h^2 = a^2 + d^2 - 2ad\cos\theta_2$$

also:

$$h^2 = b^2 + c^2 - 2bc\cos\mu$$

so:

$$a^2 + d^2 - 2ad\cos\theta_2 = b^2 + c^2 - 2bc\cos\mu$$

and:

$$\cos\mu = \frac{b^2 + c^2 - a^2 - d^2}{2bc} + \frac{ad}{bc}\cos\theta_2 \tag{4.35}$$

To find the maximum and minimum values of input angle θ_2, we can differentiate equation 4.35, form the derivative of θ_2 with respect to μ, and set it equal to zero.

$$\frac{d\theta_2}{d\mu} = \frac{bc}{ad}\frac{\sin\mu}{\sin\theta_2} = 0 \tag{4.36}$$

The link lengths a, b, c, d are never zero, so this expression can only be zero when sin μ is zero. This will be true when angle μ in Figure 4-20 is either zero or 180°. This is consistent with the definition of toggle given in Section 3.3 (p. 100). If μ is zero or

180° then cos μ will be ±1. Substituting these two values for cos μ into equation 4.35 will give a solution for the value of θ_2 between zero and 180° which corresponds to the toggle position of a triple-rocker linkage when driven from one rocker.

$$\cos\mu = \frac{b^2 + c^2 - a^2 - d^2}{2bc} + \frac{ad}{bc}\cos\theta_2 = \pm 1$$

or:

$$\cos\theta_2 = \frac{a^2 + d^2 - b^2 - c^2}{2ad} \pm \frac{bc}{ad} \tag{4.37}$$

4

and:

$$\theta_{2_{toggle}} = \arccos\left(\frac{a^2 + d^2 - b^2 - c^2}{2ad} \pm \frac{bc}{ad}\right) \qquad 0 \le \theta_{2_{toggle}} \le \pi$$

One of these ± cases will produce an argument for the arccosine function which lies between ±1. The toggle angle which is in the first or second quadrant can be found from this value. The other toggle angle will then be the negative of the one found, due to the mirror symmetry of the two toggle positions about the ground link as shown in Figure 4-16 (p. 203). Program LINKAGES computes the values of these toggle angles for any non-Grashof linkage.

4.13 CIRCUITS AND BRANCHES IN LINKAGES

In Section 4.5 (p. 183) it was noted that the fourbar linkage position problem has two solutions which correspond to the two circuits of the linkage. This section will explore the topics of circuits and branches in linkages in greater detail.

Chase and Mirth[2] define a **circuit** in a linkage as "*all possible orientations of the links that can be realized without disconnecting any of the joints*" and a **branch** as "*a continuous series of positions of the mechanism on a circuit between two stationary configurations . . . The stationary configurations divide a circuit into a series of branches.*" A linkage may have one or more circuits each of which may contain one or more branches. The number of circuits corresponds to the number of solutions possible from the position equations for the linkage.

Circuit defects are fatal to linkage operation, but branch defects are not. A mechanism that must change circuits to move from one desired position to the other (referred to as a **circuit defect**) is not useful as it cannot do so without disassembly and reassembly. A mechanism that changes branch when moving from one circuit to another (referred to as a **branch defect**) may or may not be usable depending on the designer's intent.

The tailgate linkage shown in Figure 3-2 (p. 101) is an example of a linkage with a deliberate branch defect in its range of motion (actually at the limit of its range of motion). The toggle position (stationary configuration) that it reaches with the tailgate fully open serves to hold it open. But the user can move it out of this stationary configuration by rotating one of the links out of toggle. Folding chairs and tables often use a similar scheme as do fold-down seats in automobiles.

Another example of a common linkage with a branch defect is the slider-crank linkage (crankshaft, connecting rod, and slider driving) used in every piston engine and shown

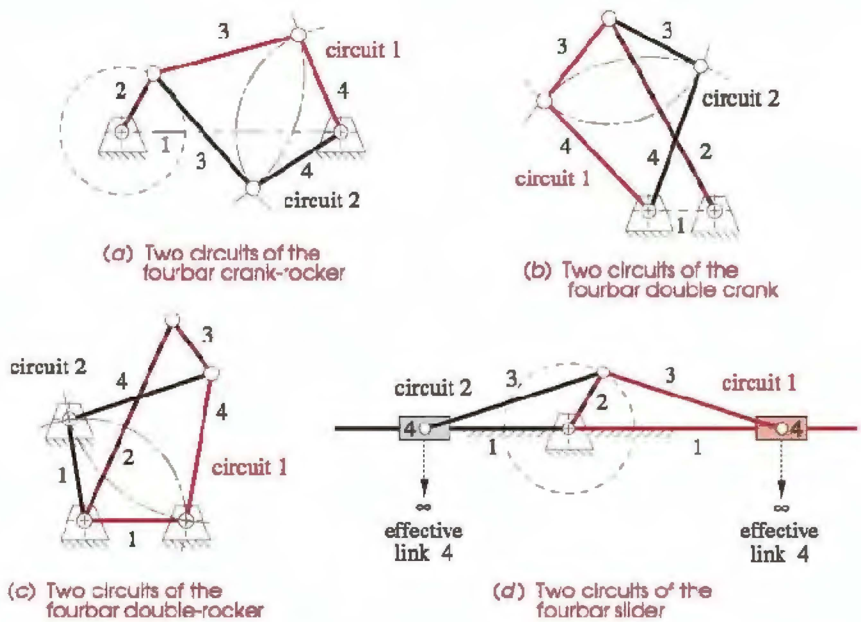


FIGURE 4-21

Circuits of the fourbar linkage

in Figure 13-3 (p. 663). This linkage has two toggle positions (top and bottom dead center) giving it two branches within one revolution of its crank. It works nevertheless because it is carried through these stationary configurations by the angular momentum of the rotating crank and its attached flywheel. One penalty is that the engine must be spun to start it in order to build sufficient momentum to carry it through these toggle positions.

The Watt sixbar linkage can have four circuits, and the Stephenson sixbar can have either four or six circuits depending on which link is driving. Eightbar linkages can have as many as 16 or 18 circuits, not all of which may be real, however.[2]

The number of circuits and branches in the fourbar linkage depends on its Grashof condition and the inversion used. A non-Grashof, triple-rocker fourbar linkage has only one circuit but has two branches. All Grashof fourbar linkages have two circuits, but the number of branches per circuit differs with the inversion. The crank-rocker and double-crank have only one branch within each circuit. The double-rocker and rocker-crank have two branches within each circuit. Table 4-1 summarizes these relationships.[2] Table 4-2 shows the circuits and branches for the two configurations of the fourbar slider linkage. Figure 4-21 shows the circuits for the Grashof fourbar linkage and the fourbar slider.

Any solution for the position of a linkage must take into account the number of possible circuits that it contains. A closed-form solution, if available, will contain all the circuits. An iterative solution such as is described in the next section will only yield the position data for one circuit, and it may not be the one you expect.

TABLE 4-1
Circuits & Branches
In the Fourbar Linkage

Fourbar Linkage Type	Number of Circuits	Branches per Circuit
Non-Grashof triple-rocker	1	2
Grashof* crank-rocker	2	1
Grashof* double-crank	2	1
Grashof* double-rocker	2	2
Grashof* rocker-crank	2	2

* Valid only for non-special-case Grashof linkages.

TABLE 4-2
Circuits & Branches
In the Fourbar Slider

Fourbar Slider Type	Number of Circuits	Branches per Circuit
Crank-slider	2	1
Slider-crank	2	2

4.14 NEWTON-RAPHSON SOLUTION METHOD

The solution methods for position analysis shown so far in this chapter are all of "closed form," meaning that they provide the solution with a direct, noniterative approach.* In some situations, particularly with multiloop mechanisms, a closed-form solution may not be attainable. Then an alternative approach is needed, and the Newton-Raphson method (sometimes just called Newton's method) provides one that can solve sets of simultaneous nonlinear equations. Any iterative solution method requires that one or more guess values be provided to start the computation. It then uses the guess values to obtain a new solution that may be closer to the correct one. This process is repeated until it converges to a solution close enough to the correct one for practical purposes. However, there is no guarantee that an iterative method will converge at all. It may diverge, taking successive solutions further from the correct one, especially if the initial guess is not sufficiently close to the real solution.

Though we will need to use the multidimensional (Newton-Raphson) version of Newton's method for these linkage problems, it is easier to understand how the algorithm works by first discussing the one-dimensional Newton method for finding the roots of a single nonlinear function in one independent variable. Then we will discuss the multidimensional Newton-Raphson method.

One-Dimensional Root-Finding (Newton's Method)

A nonlinear function may have multiple roots, where a root is defined as the intersection of the function with any straight line. Typically the zero axis of the independent variable is the straight line for which we desire the roots. Take, for example, a cubic polynomial which will have three roots, with either one or all three being real.

$$y = f(x) = -x^3 - 2x^2 + 50x + 60 \tag{4.38}$$

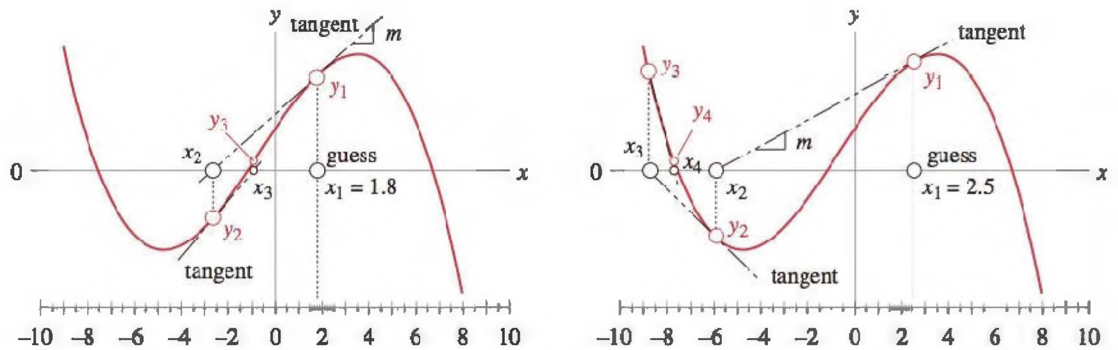
There is a closed-form solution for the roots of a cubic function† which allows us to calculate in advance that the roots of this particular cubic are all real and are $x = -7.562$, -1.177, and 6.740.

Figure 4-22 shows this function plotted over a range of x. In Figure 4-22a, an initial guess value of $x_1 = 1.8$ is chosen. Newton's algorithm evaluates the function for this guess value, finding y_1. The value of y_1 is compared to a user-selected tolerance (say 0.001) to see if it is close enough to zero to call x_1 the root. If not, then the slope (m) of the function at x_1, y_1 is calculated either by using an analytic expression for the derivative of the function or by doing a numerical differentiation (less desirable). The equation of the tangent line is then evaluated to find its intercept at x_2 which is used as a new guess value. The above process is repeated, finding y_2; testing it against the user selected tolerance; and, if it is too large, calculating another tangent line whose x intercept is used as a new guess value. This process is repeated until the value of the function y_i at the latest x_i is close enough to zero to satisfy the user.

The Newton algorithm described above can be expressed algebraically (in pseudocode) as shown in equation 4.39. The function for which the roots are sought is $f(x)$, and its derivative is $f'(x)$. The slope m of the tangent line is equal to $f'(x)$ at the current point x_i, y_i.

* Kramer [3] states that "In theory, any nonlinear algebraic system of equations can be manipulated into the form of a single polynomial in one unknown. The roots of this polynomial can then be used to determine all unknowns in the system. However, if the derived polynomial is greater than degree four, factoring and/or some form of iteration are necessary to obtain the roots. In general, systems that have more than a fourth degree polynomial associated with the eliminant of all but one variable must be solved by iteration. However, if factoring of the polynomial into terms of degree four or less is possible, all roots may be found without iteration. Therefore the only truly symbolic solutions are those that can be factored into terms of fourth degree or less. This is the formal definition of a closed form solution."

† Viete's method from "De Emendatione" by Francois Viete (1615) as described in reference [4].



(a) A guess of $x = 1.8$ converges to the root at $x = -1.177$ (b) A guess of $x = 2.5$ converges to the root at $x = -7.562$

FIGURE 4-22

Newton-Raphson method of solution for roots of nonlinear functions

step 1 $y_i = f(x_i)$

step 2 IF $y_i \le tolerance$ THEN STOP

step 3 $m = f'(x_i)$

step 4 $x_{i+1} = x_i - \dfrac{y_i}{m}$

step 5 $y_{i+1} = f(x_{i+1})$

step 6 IF $y_{i+1} \le tolerance$ THEN STOP

ELSE $x_i = x_{i+1}$: $y_i = y_{i+1}$: GOTO step 1 (4.39)

If the initial guess value is close to a root, this algorithm will converge rapidly to the solution. However, it is quite sensitive to the initial guess value. Figure 4-22b shows the result of a slight change in the initial guess from $x_1 = 1.8$ to $x_1 = 2.5$. With this slightly different guess it converges to another root. Note also that if we choose an initial guess of $x_1 = 3.579$ which corresponds to a local maximum of this function, the tangent line will be horizontal and will not intersect the x axis at all. The method fails in this situation. Can you suggest a value of x_1 that would cause it to converge to the root at $x = 6.74$?

So this method has its drawbacks. It may fail to converge. It may behave chaotically.* It is sensitive to the guess value. It also is incapable of distinguishing between multiple circuits in a linkage. The circuit solution it finds is dependent on the initial guess. It requires that the function be differentiable, and the derivative as well as the function must be evaluated at every step. Nevertheless, it is the method of choice for functions whose derivatives can be efficiently evaluated and which are continuous in the region of the root. Furthermore, it is about the only choice for systems of nonlinear equations.

* Kramer[3] points out that "the Newton Raphson algorithm can exhibit chaotic behavior when there are multiple solutions to kinematic constrain: equations. . . . Newton Raphson has no mechanism for distinguishing between the two solutions" (circuits). He does an experiment with just two links, exactly analogous to finding the angles of the coupler and rocker in the fourbar linkage position problem, and finds that the initial guess values need to be quite close to the desired solution (one of the two possible circuits) to avoid divergence or chaotic oscillation between the two solutions.

Multidimensional Root-Finding (Newton-Raphson Method)

The one-dimensional Newton method is easily extended to multiple, simultaneous, non-linear equation sets and is then called the Newton-Raphson method. First, let's generalize the expression developed for the one-dimensional case in step 4 of equation 4.39. Refer also to Figure 4-18 (p. 205).

$$x_{i+1} = x_i - \frac{y_i}{m} \qquad \text{or} \qquad m\left(x_{i+1} - x_i\right) = -y_i$$

but: $\quad y_i = f(x_i) \qquad m = f'(x_i) \qquad x_{i+1} - x_i = \Delta x$

substituting:
$$f'(x_i) \cdot \Delta x = -f(x_i) \tag{4.40}$$

4

Here a Δx term is introduced which will approach zero as the solution converges. The Δx term rather than y_i will be tested against a selected tolerance in this case. Note that this form of the equation avoids the division operation which is acceptable in a scalar equation but impossible with a matrix equation.

A multidimensional problem will have a set of equations of the form

$$\begin{bmatrix} f_1(x_1, x_2, x_3, \ldots, x_n) \\ f_2(x_1, x_2, x_3, \ldots, x_n) \\ \vdots \qquad\qquad \vdots \\ f_n(x_1, x_2, x_3, \ldots, x_n) \end{bmatrix} = \mathbf{B} \tag{4.41}$$

where the set of equations constitutes a vector, here called **B**.

Partial derivatives are required to obtain the slope terms

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = \mathbf{A} \tag{4.42}$$

which form the *Jacobian matrix* of the system, here called **A**.

The error terms are also a vector, here called **X**.

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} = \mathbf{X} \tag{4.43}$$

Equation 4.40 then becomes a matrix equation for the multidimensional case.

$$\mathbf{AX} = -\mathbf{B} \tag{4.44}$$

Equation 4.44 can be solved for **X** either by matrix inversion or by Gaussian elimination. The values of the elements of **A** and **B** are calculable for any assumed (guess) values of the variables. A criterion for convergence can be taken as the sum of the error vector **X** at each iteration where the sum approaches zero at a root.

Let's set up this Newton-Raphson solution for the fourbar linkage.

Newton-Raphson Solution for the Fourbar Linkage

The vector loop equation of the fourbar linkage, separated into its real and imaginary parts (equations 4.6a and 4.6b, p. 187) provides the set of functions that define the two unknown link angles θ_3 and θ_4. The link lengths, *a, b, c, d*, and the input angle θ_2 are given.

$$f_1 = a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d = 0$$
$$f_2 = a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 = 0 \tag{4.45a}$$

$$\mathbf{B} = \begin{bmatrix} a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d \\ a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 \end{bmatrix} \tag{4.45b}$$

The error vector is:

$$\mathbf{X} = \begin{bmatrix} \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} \tag{4.46}$$

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial f_1}{\partial\theta_3} & \dfrac{\partial f_1}{\partial\theta_4} \\ \dfrac{\partial f_2}{\partial\theta_3} & \dfrac{\partial f_2}{\partial\theta_4} \end{bmatrix} = \begin{bmatrix} -b\sin\theta_3 & c\sin\theta_4 \\ b\cos\theta_3 & -c\cos\theta_4 \end{bmatrix} \tag{4.47}$$

This matrix is known as the **Jacobian** of the system, and, in addition to its usefulness in this solution method, it also tells something about the solvability of the system. The system of equations for position, velocity, and acceleration (in all of which the Jacobian appears) can only be solved if the value of the determinant of the Jacobian is nonzero.

Substituting equations 4.45b, 4.46, and 4.47 into equation 4.44 gives:

$$\begin{bmatrix} -b\sin\theta_3 & c\sin\theta_4 \\ b\cos\theta_3 & -c\cos\theta_4 \end{bmatrix} \begin{bmatrix} \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} = -\begin{bmatrix} a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d \\ a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 \end{bmatrix} \tag{4.48}$$

To solve this matrix equation, guess values will have to be provided for θ_3 and θ_4 and the two equations then solved simultaneously for $\Delta\theta_3$ and $\Delta\theta_4$. For a larger system of equations, a matrix reduction algorithm will need to be used. For this simple system in two unknowns, the two equations can be solved by combination and reduction. The test described above which compares the sum of the values of $\Delta\theta_3$ and $\Delta\theta_4$ to a selected tolerance must be applied after each iteration to determine if a root has been found.

Equation Solvers

Some commercially available equation solver software packages include the ability to do a Newton-Raphson iterative solution on sets of nonlinear simultaneous equations. *TKSolver** and *Mathcad*† are examples. *TKSolver* automatically invokes its Newton-

* Universal Technical Systems, 1220 Rock St. Rockford, IL 61101, USA. (800) 435-7887

† PTC Inc., 140 Kendrick St., Needham, MA 02494 (781) 370-5000

TABLE P4-0
Topic/Problem Matrix

4.2 Position and Displacement
4-53, 4-57

4.5 Position Analysis of Fourbar Linkages
4-1, 4-2, 4-3, 4-4, 4-5
Graphical 4-6
Analytical 4-7, 4-8, 4-18d, 4-24, 4-36, 4-39, 4-42, 4-45, 4-48, 4-51, 4-58, 4-59

4.6 Fourbar Crank-Slider Position Solution
Graphical 4-9
Analytical 4-10, 4-18c, 4-18f, 4-18h, 4-20

4.7 Fourbar Slider-Crank Position Solution
Graphical 4-60
Analytical 4-61

4.8 Inverted Crank-Slider Position Solution
Graphical 4-11
Analytical 4-12, 4-48

4.9 Linkages of More than Four Bars
Graphical GFBM 4-16
Analytical GFBM 4-17
Sixbar 4-34, 4-36, 4-37, 4-39, 4-40, 4-42, 4-49, 4-51
Eightbar 4-43, 4-45

4.10 Position of Any Point on a Linkage
4-19, 4-22, 4-23, 4-46

4.11 Transmission Angles
4-13, 4-14, 4-18b, 4-18e, 4-35, 4-38, 4-41, 4-44, 4-47, 4-50, 4-54

4.12 Toggle Positions
4-15, 4-18a, 4-18g, 4-21, 4-25, 4-26, 4-27, 4-28, 4-29, 4-30, 4-52, 4-55, 4-56

4.14 Newton-Raphson Solution Method
4-31, 4-32, 4-33

Raphson solver when it cannot directly solve the presented equation set, provided that enough guess values have been supplied for the unknowns. These equation solver tools are quite convenient in that the user need only supply the equations for the system in "raw" form such as equation 4.45a . It is not necessary to arrange them into the Newton-Raphson algorithm as shown in the previous section. Lacking such a commercial equation solver, you will have to write your own computer code to program the solution as described above. Reference [5] is a useful aid in this regard. The DVD included with this text contains example *TKSolver* files for the solution of this fourbar position problem as well as others.

4.15 REFERENCES

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5 Ibid., pp. 254-273.

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4.16 PROBLEMS‡

4-1 A position vector is defined as having a length equal to your height in inches (or centimeters). The tangent of its angle is defined as your weight in pounds (or kilograms) divided by your age in years. Calculate the data for this vector and:

a. Draw the position vector to scale on cartesian axes.
b. Write an expression for the position vector using unit vector notation.
c. Write an expression for the position vector using complex number notation, in both polar and cartesian forms.

4-2 A particle is traveling along an arc of 6.5-in radius. The arc center is at the origin of a coordinate system. When the particle is at position *A*, its position vector makes a

‡ All problem figures are provided as PDF files, and some are also provided as animated AVI and Working Model files; all are on the DVD. PDF filenames are the same as the figure number. Run the file *Animations.html* to access and run the animations.