VELOCITY ANALYSIS

The faster I go, the behinder I get ANON. PENN. DUTCH

6.0 INTRODUCTION

Once a position analysis is done, the next step is to determine the velocities of all links and points of interest in the mechanism. We need to know the velocities in our mechanism or machine, both to calculate the stored kinetic energy from $mV^2/2$, and also as a step on the way to the determination of the link's accelerations which are needed for the dynamic force calculations. Many methods and approaches exist to find velocities in mechanisms. We will examine only a few of these methods here. We will first develop manual graphical methods, which are often useful as a check on the more complete and accurate analytical solution. We will also investigate the properties of the instant center of velocity which can shed much light on a mechanism's velocity behavior with very little effort. Finally, we will derive the analytical solution for the fourbar and inverted slider-crank as examples of the general vector loop equation solution to velocity analysis problems. From these calculations we will be able to establish some indices of merit to judge our designs while they are still on the drawing board (or in the computer).

6.1 DEFINITION OF VELOCITY

Velocity is defined as *the rate of change of position with respect to time*. Position (**R**) is a vector quantity and so is velocity. Velocity can be **angular** or **linear**. **Angular velocity** will be denoted as ω and **linear velocity** as **V**.

$$\omega = \frac{d\theta}{dt}; \qquad \mathbf{V} = \frac{d\mathbf{R}}{dt} \tag{6.1}$$

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Figure 6-1 shows a link *PA* in pure rotation, pivoted at point *A* in the *xy* plane. Its position is defined by the position vector \mathbf{R}_{PA} . We are interested in the velocity of point



FIGURE 6-1

A link in pure rotation

P when the link is subjected to an angular velocity ω . If we represent the position vector \mathbf{R}_{PA} as a complex number in polar form,

$$\mathbf{R}_{PA} = p e^{j \theta} \tag{6.2}$$

where p is the scalar length of the vector. We can easily differentiate it to obtain:

$$\mathbf{V}_{PA} = \frac{d\mathbf{R}_{PA}}{dt} = p j e^{j\theta} \frac{d\theta}{dt} = p \omega j e^{j\theta}$$
(6.3)

Compare the right side of equation 6.3 to the right side of equation 6.2. Note that as a result of the differentiation, the velocity expression has been multiplied by the (constant) complex operator *j*. This causes a rotation of this velocity vector through 90 degrees with respect to the original position vector. (See also Figure 4-5b, p. 152.) This 90-degree rotation is positive, or counterclockwise. However, the velocity expression is also multiplied by ω , which may be either positive or negative. As a result, the velocity vector will be **rotated 90 degrees** from the angle θ of the position vector **in a direction dictated by the sign of** ω . This is just mathematical verification of what you already knew, namely that *velocity is always in a direction perpendicular to the radius of rotation and is tangent to the path of motion* as shown in Figure 6-1.

Substituting the Euler identity (equation 4.4a, p. 155) into equation 6.3 gives us the real and imaginary (or x and y) components of the velocity vector.

$$\mathbf{V}_{PA} = p\,\omega\,j(\cos\theta + j\sin\theta) = p\,\omega(-\sin\theta + j\cos\theta) \tag{6.4}$$

Note that the sine and cosine terms have swapped positions between the real and imaginary terms, due to multiplying by the *j* coefficient. This is evidence of the 90 degree rotation of the velocity vector versus the position vector. The former *x* component has become the *y* component, and the former *y* component has become a minus *x* component. Study Figure 4-5b (p. 152) to review why this is so.

The velocity V_{PA} in Figure 6-1 can be referred to as an **absolute velocity** since it is referenced to *A*, which is the origin of the global coordinate axes in that system. As such, we could have referred to it as V_P , with the absence of the second subscript implying reference to the global coordinate system. Figure 6-2a shows a different and slightly







more complicated system in which the pivot A is no longer stationary. It has a known linear velocity V_A as part of the translating carriage, link 3. If ω is unchanged, the velocity of point P versus A will be the same as before, but V_{PA} can no longer be considered an absolute velocity. It is now a velocity difference and must carry the second subscript as V_{PA} . The absolute velocity V_P must now be found from the velocity difference equation whose graphical solution is shown in Figure 6-2b:

$$\mathbf{V}_{PA} = \mathbf{V}_P - \mathbf{V}_A \tag{6.5a}$$

rearranging:

$$\mathbf{V}_P = \mathbf{V}_A + \mathbf{V}_{PA} \tag{6.5b}$$

Note the similarity of equation 6.5 to the position difference equation 4.1 (p. 147).

Figure 6-3 shows two independent bodies P and A, which could be two automobiles, moving in the same plane. If their independent velocities V_P and V_A are known, their relative velocity V_{PA} can be found from equation 6.5 arranged algebraically as:

$$\mathbf{V}_{PA} = \mathbf{V}_P - \mathbf{V}_A \tag{6.6}$$

The graphical solution to this equation is shown in Figure 6-3b. Note that it is similar to Figure 6-2b except for a different vector being the resultant.

As we did for position analysis, we give these two cases different names despite the **fact that** the same equation applies. Repeating the definition from Section 4.2 (p. 147), **modified** to refer to velocity:

CASE 1: Two points in the same body => velocity difference

CASE 2: Two points in different bodies => relative velocity

We will find use for this semantic distinction both when we analyze linkage velocities and the velocity of slip later in this chapter.





Relative velocity

6.2 GRAPHICAL VELOCITY ANALYSIS

Before programmable calculators and computers became universally available to engineers, graphical methods were the only practical way to solve these velocity analysis problems. With some practice and with proper tools such as a drafting machine or *CAD* package, one can fairly rapidly solve for the velocities of particular points in a mechanism for any one input position by drawing vector diagrams. However, it is a tedious process if velocities for many positions of the mechanism are to be found, because each new position requires a completely new set of vector diagrams be drawn. Very little of the work done to solve for the velocities at position 1 carries over to position 2, etc. Nevertheless, this method still has more than historical value as it can provide a quick check on the results from a computer program solution. Such a check needs only be done for a few positions to prove the validity of the program. Also, graphical solutions provide the beginning student some visual feedback on the solution which can help develop an understanding of the underlying principles. It is principally for this last reason that graphical solutions are included in this text even in this "age of the computer."

To solve any velocity analysis problem graphically, we need only two equations, 6.5 and 6.7 (which is merely the scalar form of equation 6.3):

$$|\mathbf{V}| = v = r\omega \tag{6.7}$$

Note that the scalar equation 6.7 defines only the **magnitude** (v) of the velocity of any point on a body which is in pure rotation. In a graphical CASE 1 analysis, the **direction** of the vector due to the rotation component must be understood from equation 6.3 to be perpendicular to the radius of rotation. Thus, if the center of rotation is known, the direction of the velocity component due to that rotation is known and its sense will be consistent with the angular velocity ω of the body.



(b) Velocity diagram for points A and B



(c) Velocity diagram for points A and C



(a) Linkage showing velocity of point A



(d) Linkage showing velocities of points A, B, and C

FIGURE 6-4

Graphical solution for velocities in a pin-jointed linkage

Figure 6-4 shows a fourbar linkage in a particular position. We wish to solve for the angular velocities of links 3 and 4 (ω_3 , ω_4) and the linear velocities of points *A*, *B*, and *C* (\mathbf{V}_A , \mathbf{V}_B , \mathbf{V}_C). Point *C* represents any general point of interest. Perhaps *C* is a coupler point. The solution method is valid for any point on any link. To solve this problem we need to know the *lengths of all the links*, the *angular positions of all the links*, and the *instantaneous input velocity of any one driving link or driving point*. Assuming we have designed this linkage, we will know or can measure the link lengths. We must also first do a **complete position analysis** to find the link angles θ_3 and θ_4 given the input link's position θ_2 . This can be done by any of the methods in Chapter 4. In general we must solve these problems in stages, first for link positions, then for velocities, and finally for accelerations. For the following example, we will assume that a complete position analysis has been done and that the input is to link 2 with known θ_2 and ω_2 for this one "freeze frame" position of the moving linkage.

EDEXAMPLE 6-1

Graphical Velocity Analysis for One Position of Linkage.

Problem: Given θ_2 , θ_3 , θ_4 , ω_2 , find ω_3 , ω_4 , V_A , V_B , V_C by graphical methods.

Solution: (see Figure 6-4, p. 245)

1 Start at the end of the linkage about which you have the most information. Calculate the magnitude of the velocity of point *A* using scalar equation 6.7.

$$v_a = (AO_2)\omega_2 \tag{a}$$

- 2 Draw the velocity vector \mathbf{V}_A with its length equal to its magnitude v_A at some convenient scale with its root at point A and its direction perpendicular to the radius AO_2 . Its sense is the same as that of ω_2 as shown in Figure 6-4a.
- 3 Move next to a point about which you have some information. Note that the direction of the velocity of point *B* is predictable since it is pivoting in pure rotation about point O_4 . Draw the construction line *pp* through point *B* perpendicular to BO_4 , to represent the direction of V_B as shown in Figure 6-4a.
- 4 Write the velocity difference vector equation 6.5 for point B versus point A.

$$\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA} \tag{b}$$

We will use point A as the reference point to find VB because A is in the same link as Band we have already solved for VA- Any vector equation can be solved for two unknowns. Each term has two parameters, namely magnitude and direction. There are then potentially six unknowns in this equation, two per term. We must know four of them to solve it. We know both magnitude and direction of VA and the direction of VB' We need to know one more parameter.

- S The term VBA represents the velocity of B with respect to A. If we assume that the link BA is rigid, then there can be no component of VBA which is directed along the line BA, because point B cannot move toward or away from point A without shrinking or stretching the rigid link! Therefore, the direction of VBA must be perpendicular to the line BA. Draw construction line qq through point B and perpendicular to BA to represent the direction of VBA, as shown in Figure 6-4a.
- 6 Now the vector equation can be solved graphically by drawing a vector diagram as shown in Figure 6-4b. Either drafting tools or a CAD package is necessary for this step. First draw velocity vector V_A carefully to some scale, maintaining its direction. (It is drawn twice size in the figure.) The equation in step 4 says to add VBA to VA, so draw a line parallel to line qq across the tip of VA' The resultant, or left side of the equation, must close the vector diagram, from the tail of the first vector drawn (VA) to the tip of the last, so draw a line parallel to pp across the tail of VA' The intersection of these lines parallel to pp and qq defines the lengths of VB and VBA. The senses of the vectors are determined from reference to the equation. VA was added to VBA, so they must be arranged tip to tail. VB is the resultant, so it must be from the tail of the first to the tip of the last. The resultant vectors are shown in Figure 6-4b and d.

7 The angular velocities of links 3 and 4 can be calculated from equation 6.7:

$$\omega_4 = \frac{v_B}{BO_4}$$
 and $\omega_3 = \frac{v_{BA}}{BA}$ (c)

Note that the velocity difference term V_{BA} represents the rotational component of velocity of link 3 due to ω_3 . This must be true if point *B* cannot move toward or away from point *A*. The only velocity difference they can have, one to the other, is due to rotation of the line connecting them. You may think of point *B* on the line *BA* rotating about point *A* as a center, or point *A* on the line *AB* rotating about *B* as a center. The rotational velocity ω of any body is a "free vector" which has no particular point of application to the body.

8 Finally we can solve for V_{C_i} again using equation 6.5. We select any point in link 3 for which we know the absolute velocity to use as the reference, such as point A.

$$\mathbf{V}_C = \mathbf{V}_A + \mathbf{V}_{CA} \tag{d}$$

In this case, we can calculate the magnitude of V_{CA} from equation 6.7 as we have already found ω_3 ,

$$v_{ca} = c\omega_3 \tag{e}$$

Since both V_A and V_{CA} are known, the vector diagram can be directly drawn as shown in Figure 6-4c. V_C is the resultant which closes the vector diagram. Figure 6-4d shows the calculated velocity vectors on the linkage diagram. Note that the velocity difference vector V_{CA} is perpendicular to line *CA* (along line *rr*) for the same reasons as discussed in step 7 above.

The above example contains some interesting and significant principles which deserve further emphasis. Equation 6.5a is repeated here for discussion.

$$\mathbf{V}_P = \mathbf{V}_A + \mathbf{V}_{PA} \tag{6.5a}$$

This equation represents the *absolute* velocity of some general point P referenced to the origin of the global coordinate system. The right side defines it as the sum of the absolute velocity of some other reference point A in the same system and the velocity difference (or relative velocity) of point P versus point A. This equation could also be written:

Velocity = translation component + rotation component

These are the same two components of motion defined by Chasles' theorem, and introduced for displacement in Section 4.3 (p. 149). Chasles' theorem holds for velocity as well. These two components of motion, translation, and rotation, are independent of one another. If either is zero in a particular example, the complex motion will reduce to one of the special cases of pure translation or pure rotation. When both are present, the total velocity is merely their vector sum.

Let us review what was done in Example 6-1 in order to extract the general strategy for solution of this class of problem. We started at the input side of the mechanism, as that is where the driving angular velocity is defined. We first looked for a point (A) for

which the motion was pure rotation so that one of the terms in equation 6.5 (p. 243) would be zero. (We could as easily have looked for a point in pure translation to boot-strap the solution.) We then solved for the absolute velocity of that point (VA) using equations 6.5 and 6.7 (p. 244). (Steps I and 2)

We then used the point (A) just solved for as a reference point to define the translation component in equation 6.5 written for a new point (B). Note that we needed to choose a second point (B) which was in the same rigid body as the reference point (A) which we had already solved and about which we could predict some aspect of the new point's (B's) velocity. In this example, we knew the direction of the velocity VB. In general this condition will be satisfied by any point on a link which is jointed to ground (as is link 4). In this example, we could not have solved for point C until we solved for B, because point C is on a floating link for which point we do not yet know the velocity direction. (Steps 3 and 4)

To solve the equation for the second point (B), we also needed to recognize that the rotation component of velocity is directed perpendicular to the line connecting the two points in the link (B and A in the example). You will always know the direction of the rotation component in equation 6.5 if it represents a velocity difference (CASE 1) situation. If the rotation component relates two points in the same rigid body, then that velocity difference component is always perpendicular to the line connecting those two points (see Figure 6-2, p. 243). This will be true regardless of the two points selected. But, this is not true in a CASE 2 situation (see Figure 6-3, p. 244). (Steps 5 and 6)

Once we found the absolute velocity (VB) of a second point on the same link (CASE 1) we could solve for the angular velocity of that link. (Note that points A and *Bare* both on link 3 and the velocity of point 04 is zero.) Once the angular velocities of all the links were known, we could solve for the linear velocity of any point (such as C) in any link using equation 6.5. To do so, we had to understand the concept of angular velocity as a free vector, meaning that it exists everywhere on the link at any given instant. It has no particular center. It has an infinity of potential centers. The link simply has an angular velocity, just as does a frisbee thrown and spun across the lawn.

All points on *afrisbee*, if spinning while flying, obey equation 6.5. Left to its own devices, the frisbee will spin about its center of gravity (*CG*), which is close to the center of its circular shape. But if you are an expert frisbee player (and have rather pointed fingers), you can imagine catching that flying frisbee between your two index fingers in some off-center location (not at the *CG*), such that the frisbee continues to spin about your fingertips. In this somewhat far-fetched example of championship frisbee play, you will have taken the translation component of the frisbee's motion to zero, but its independent rotation component will still be present. Moreover, it will now be spinning about a different center (your fingers) than it was in flight (its *CG*). Thus this free vector of angular velocity $\ll(0)$ is happy to attach itself to any point on the body. The body still has the same 00, regardless of the assumed center of rotation. It is this property that allows us to solve equation 6.5 for literally any point on a rigid body in complex motion referenced to any other point on that body. (*Steps 7 and 8*).

6.3 INSTANT CENTERS OF VELOCITY

The definition of an **instant center** of velocity is *a point, common to two bodies in plane motion, which point has the same instantaneous velocity in each body.* Instant centers are sometimes also called *centros* or *poles*. Since it takes two bodies or links to create an instant center (IC), we can easily predict the quantity of instant centers to expect from any collection of links. The combination formula for *n* things taken *r* at a time is:

$$C = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$
(6.8a)

For our case r = 2 and it reduces to:

$$C = \frac{n(n-1)}{2} \tag{6.8b}$$

From equation 6.8b we can see that a fourbar linkage has 6 instant centers, a sixbar has 15, and an eightbar has 28.

Figure 6-5 shows a fourbar linkage in an arbitrary position. It also shows a **linear** graph * which is useful for keeping track of which lCs have been found. This particular graph can be created by drawing a circle on which we mark off as many points as there are links in our assembly. We will then draw a line between the dots representing the link pairs each time we find an instant center. The resulting linear graph is the set of lines connecting the dots. It does not include the circle which was used only to place the dots. This graph is actually a geometric solution to equation 6.8b, since connecting all the points in pairs gives all the possible combinations of points taken two at a time.

Some lCs can be found by inspection, using only the definition of the instant center. Note in Figure 6-5a that the four pin joints each satisfy the definition. They clearly must have the same velocity in both links at all times. These have been labeled h, 2, h, 3, h, 4, and h, 4' The order of the subscripts is immaterial. Instant center h, l is the same as h, 2. These pin joint lCs are sometimes called "permanent" instant centers as they remain in the same location for all positions of the linkage. In general, instant centers will move to new locations as the linkage changes position, thus the adjective *instant*. In this fourbar example there are two more lCs to be found. It will help to use the Aronhold-Kennedy theorem,t also called *Kennedy's rule*, to locate them.

Kennedy's rule:

Any three bodies in plane motion will have exactly three instant centers, and they will lie on the same straight line.

The first part of this rule is just a restatement of equation 6.8b for n = 3. It is the second clause in this rule that is most useful. Note that this rule does **not** require that the three bodies be connected in any way. We can use this rule, in conjunction with the linear graph, to find the remaining *lCs* which are not obvious from inspection. Figure 6.5b shows the construction necessary to find instant center 11,3. Figure 6-5c shows the construction necessary to find instant center h, 4' The following example describes the procedure in detail.

* Note that this graph is not a plot of points on an x, y coordinate system. Rather it is a linear graph from the fascinating branch of mathematics called graph theory, which is itself a branch of topology. Linear graphs are often used to depict interrelationships between various phenomena. They have many applications in kinematics especially as a way to classify linkages and to find isomers.

[†] Discovered independently by Aronhold in Germany, in 1872, and by Kennedy in England, in 1886.

EDEXAMPLE 6-2

Finding All Instant Centers for a Fourbar Linkage.

Problem: Given a fourbar linkage in one position, find all *ICs* by graphical methods.

Solution: (see Figure 6-5)

- 1 Draw a circle with all links numbered around the circumference as shown in Figure 6-5a.
- ² Locate as many *ICs* as possible by inspection. All pin joints will be permanent *ICs*. Connect the link numbers on the circle to create a linear graph and record those *ICs* found, as shown in Figure 6-5a.
- 3 Identify a link combination on the linear graph for which the *IC* has not been found, and draw a dotted line connecting those two link numbers. Identify two triangles on the graph which each contain the dotted line and whose other two sides are solid lines representing *ICs* already found. On the graph in Figure 6-5b, link numbers 1 and 3 have been connected with a dotted line. This line forms one triangle with sides 13, 34, 14 and another with sides 13, 23, 12. These triangles define trios of *ICs* which obey **Kennedy's rule**. Thus *ICs* 13, 34, and 14 **must lie on the same straight line**. Also *ICs* 13, 23 and 12 will **lie on a different straight line**.
- 4 On the linkage diagram draw a line through the two known *ICs* which form a trio with the unknown *IC*. Repeat for the other trio. In Figure 6-5b, a line has been drawn through $I_{1,2}$ and $I_{2,3}$ and extended. $I_{1,3}$ must lie on this line. Another line has been drawn through $I_{1,4}$ and $I_{3,4}$ and extended to intersect the first line. By Kennedy's rule, instant center $I_{1,3}$ must also lie on this line, so their intersection is $I_{1,3}$.
- ⁵ Connect link numbers 2 and 4 with a dotted line on the linear graph as shown in Figure 6-5c. This line forms one triangle with sides 24, 23, 34 and another with sides 24, 12, 14. These sides represent trios of *ICs* which obey Kennedy's rule. Thus *ICs* 24, 23, and 34 must lie on the same straight line. Also *ICs* 24, 12, and 14 lie on a different straight line.
- 6 On the linkage diagram draw a line through the two known *ICs* which form a trio with the unknown *IC*. Repeat for the other trio. In Figure 6-5c, a line has been drawn through $I_{1,2}$ and $I_{1,4}$ and extended. $I_{2,4}$ must lie on this line. Another line has been drawn through $I_{2,3}$ and $I_{3,4}$ and extended to intersect the first line. By Kennedy's rule, instant center $I_{2,4}$ must also lie on this line, so their intersection is $I_{2,4}$.
- 7 If there were more links, this procedure would be repeated until all ICs were found.

The presence of slider joints makes finding the instant centers a little more subtle as is shown in the next example. Figure 6-6a shows a **fourbar slider-crank linkage**. Note that there are only three pin joints in this linkage. All pin joints are *permanent instant centers*. But the joint between links I and 4 is a rectilinear, sliding full joint. A sliding joint is kinematically equivalent to an infinitely long link, "pivoted" at infinity. Figure 6-6b shows a nearly equivalent pin-jointed version of the slider-crank in which link 4 is a very long rocker. Point *B* now swings through a shallow are which is nearly a straight





line. It is clear in Figure 6-6b that, in this linkage, h,4 is at pivot 04. Now imagine increasing the length of this long, link 4 rocker even more. In the limit, link 4 approaches infinite length, the pivot 04 approaches infinity along the line which was originally the long rocker, and the arc motion of point *B* approaches a straight line. Thus, *a slider joint will have its instant center at infinity along a line perpendicular to the direction of sliding* as shown in Figure 6-6a.



FIGURE 6-6

A rectilinear slider's instant center is at infinity

EIEXAMPLE 6-3

Finding All Instant Centers for a Slider-Crank Linkage.

Problem: Given a slider-crank linkage in one position, find all *ICs* by graphical methods.

Solution: (see Figure 6-7)

- 1 Draw a circle with all links numbered around the circumference as shown in Figure 6-7a.
- 2 Locate all *ICs* possible by inspection. All pin joints will be permanent *ICs*. The slider joint's instant center will be at infinity along a line perpendicular to the axis of sliding. Connect the link numbers on the circle to create a linear graph and record those *ICs* found, as shown in Figure 6-7a.
- 3 Identify a link combination on the linear graph for which the *IC* has not been found, and draw a dotted line connecting those two link numbers. Identify two triangles on the graph which each contain the dotted line and whose other two sides are solid lines representing *ICs* already found. In the graph on Figure 6-7b, link numbers 1 and 3 have been connected with a dotted line. This line forms one triangle with sides 13, 34, 14 and another with sides 13, 23, 12. These sides represent trios of *ICs* which obey Kennedy's rule. Thus *ICs* 13, 34, and 14 must lie on the same straight line. Also *ICs* 13, 23, and 12 lie on a different straight line.
- 4 On the linkage diagram draw a line through the two known *ICs* which form a trio with the unknown *IC*. Repeat for the other trio. In Figure 6-7b, a line has been drawn from $I_{1,2}$. through $I_{2,3}$ and extended. $I_{1,3}$ must lie on this line. Another line has been drawn from $I_{1,4}$



FIGURE 6-7

Locating instant centers in the slider-crank linkage

(at infinity) through $I_{3,4}$ and extended to intersect the first line. By Kennedy's rule, instant center $I_{1,3}$ must also lie on this line, so their intersection is $I_{1,3}$.

5 Connect link numbers 2 and 4 with a dotted line on the graph as shown in Figure 6-7c. This line forms one triangle with sides 24, 23, 34 and another with sides 24, 12, 14. These sides also represent trios of *ICs* which obey Kennedy's rule. Thus *ICs* 24, 23, and 34 must lie on the same straight line. Also *ICs* 24, 12, and 14 lie on a different straight line.

6 On the linkage diagram draw a line through the two known *ICs* which form a trio with the unknown *IC*. Repeat for the other trio. In Figure 6-7c, a line has been drawn from $I_{1,2}$ to intersect $I_{1,4}$, and extended. Note that the only way to "intersect" $I_{1,4}$ at infinity is to draw a line parallel to the line $I_{3,4}I_{1,4}$ since all parallel lines intersect at infinity. Instant center $I_{2,4}$ must lie on this parallel line. Another line has been drawn through $I_{2,3}$ and $I_{3,4}$ and extended to intersect the first line. By Kennedy's rule, instant center $I_{2,4}$ must also lie on this line, so their intersection is $I_{2,4}$.

7 If there were more links, this procedure would be repeated until all *ICs* were found.

The procedure in this slider example is identical to that used in the pin-jointed fourbar, except that it is complicated by the presence of instant centers located at infinity.

In Section 2.9 and Figure 2-10c (p. 41) we showed that a cam-follower mechanism is really a fourbar linkage in disguise. As such it will also possess instant centers. The presence of the half joint in this, or any linkage, makes the location of the instant centers a little more complicated. We have to recognize that the instant center between any two links will be along a line that is perpendicular to the *relative velocity* vector between the links at the half joint, as shown in the following example. Figure 6-8 shows the same cam-follower mechanism as in Figure 2-14 (p. 45). The effective links 2, 3, and 4 are also shown.

EDEXAMPLE 6-4

Finding All Instant Centers for a Cam-Follower Mechanism.

Problem: Given a cam and follower in one position, find all *ICs* by graphical methods.

Solution: (see Figure 6-8)

- 1 Draw a circle with all links numbered around the circumference as shown in Figure 6-8b. In this case there are only three links and thus only three *ICs* to be found as shown by equation 6.8. Note that the links are numbered 1, 2, and 4. The missing link 3 is the variable-length effective coupler.
- ² Locate all *ICs* possible by inspection. All pin joints will be permanent *ICs*. The two fixed pivots $I_{1,2}$ and $I_{1,4}$ are the only pin joints here. Connect the link numbers on the circle to create a linear graph and record those *ICs* found, as shown in Figure 6-8b. The only link combination on the linear graph for which the *IC* has not been found is $I_{2,4}$, so draw a dotted line connecting those two link numbers.
- 3 Kennedy's rule says that all three *ICs* must lie on the same straight line; thus the remaining instant center $I_{2,4}$ must lie on the line $I_{1,2} I_{1,4}$ extended. Unfortunately in this example, we have too few links to find a second line on which $I_{2,4}$ must lie.
- 4 On the linkage diagram draw a line through the two known ICs which form a trio with the unknown IC. In Figure 6-8c, a line has been drawn from $I_{1,2}$ through $I_{1,4}$ and extended. This is, of course, link 1. By Kennedy's rule, $I_{2,4}$ must lie on this line.



FIGURE 6-8

Locating instant centers in the cam-follower mechanism

- 5 Looking at Figure 6-8c which shows the effective links of the equivalent fourbar linkage for this position, we can extend effective link 3 until it intersects link 1 extended. Just as in the "pure" fourbar linkage, instant center 2,4 lies on the intersection of links 1 and 3 extended (see Example 6-2, p. 250).
- 6 Figure 6-8d shows that it is not necessary to construct the effective fourbar linkage to find $I_{2,4}$. Note that the **common tangent** to links 2 and 4 at their contact point (the half joint) has been drawn. This line is also called the **axis of slip** because it is the line along which all relative (slip) velocity will occur between the two links. Thus the velocity of link 4 versus 2, V_{42} , is directed along the axis of slip. Instant center $I_{2,4}$ must therefore lie along a line perpendicular to the common tangent, called the **common normal.** Note that this line is the same as the effective link 3 line in Figure 6-8c.

6.4 VELOCITY ANALYSIS WITH INSTANT CENTERS

Once the *ICs* have been found, they can be used to do a very rapid graphical velocity analysis of the linkage. Note that, depending on the particular position of the linkage being analyzed, some of the *ICs* may be very far removed from the links. For example, if links 2 and 4 are nearly parallel, their extended lines will intersect at a point far away and not be practically available for velocity analysis. Figure 6-9 shows the same linkage as Figure 6-5 (p. 251) with $I_{1,3}$ located and labeled. From the definition of the instant center, both links sharing the instant center will have identical velocity at that point. Instant center $I_{1,3}$ involves the coupler (link 3) which is in complex motion, and the ground link 1, which is stationary. All points on link 1 have zero velocity in the global coordinate system, which is embedded in link 1. Therefore, $I_{1,3}$ must have zero velocity at this instant. If $I_{1,3}$ has zero velocity, then it can be considered to be an instantaneous "fixed pivot" about which link 3 is in pure rotation with respect to link 1. A moment later, $I_{1,3}$ will move to a new location and link 3 will be "pivoting" about a new instant center.

The velocity of point *A* is shown on Figure 6-9. The magnitude of V_A can be computed from equation 6.7 (p. 244). Its direction and sense can be determined by inspection as was done in Example 6-1 (p. 246). Note that point *A* is also instant center $I_{2,3}$. It has the same velocity as part of link 2 and as part of link 3. Since link 3 is effectively pivoting about $I_{1,3}$ at this instant, the angular velocity ω_3 can be found by rearranging equation 6.7:

$$\omega_3 = \frac{\nu_A}{\left(AI_{1,3}\right)} \tag{6.9a}$$

Once w_3 is known, the magnitude of V_B can also be found from equation 6.7:

$$v_B = (BI_{1,3})\omega_3 \tag{6.9b}$$

Once V_B is known, w_4 can also be found from equation 6.7:

$$\omega_4 = \frac{v_B}{(BO_4)} \tag{6.9c}$$

Finally, the magnitude of V_C (or the velocity of any other point on the coupler) can be found from equation 6.7:

$$v_C = (CI_{1,3})\omega_3 \tag{6.9d}$$

Note that equations 6.7 and 6.9 provide only the **scalar magnitude** of these velocity vectors. We have to determine their **direction** from the information in the scale diagram (Figure 6-9). Since we know the location of $I_{1,3}$, which is an instantaneous "fixed" pivot for link 3, all of that link's absolute velocity vectors for this instant will be **perpendicular to their radii from I**_{1,3}. Note that V_B is also perpendicular to the radius from O_4 because *B* is also pivoting about that point as part of link 4.

A rapid graphical solution to equations 6.9 is shown in the figure. Arcs centered at $I_{1,3}$ are swung from points B and C to intersect line $AI_{1,3}$. The magnitudes of velocities



FIGURE 6-9

Velocity analysis using instant centers

 $V_{B'}$ and $V_{C'}$ are found from the vectors drawn perpendicular to that line at the intersections of the arcs and line $AI_{1,3}$. The lengths of the vectors are defined by the line from the tip of V_A to the instant center $I_{1,3}$. These vectors can then be slid along their arcs back to points *B* and *C*, maintaining their tangency to the arcs.

Thus, we have in only a few steps found all the same velocities that were found in the more tedious method of Example 6-1. The instant center method is a quick graphical method to analyze velocities, but it will only work if the instant centers are in reachable locations for the particular linkage position analyzed. However, the graphical method using the velocity difference equation shown in Example 6-1 will always work, regardless of linkage position.

Angular Velocity Ratio

The **angular velocity ratio** *mV* is defined as *the output angular velocity divided by the input angular velocity*. For a fourbar mechanism this is expressed as:

$$m_V = \frac{\omega_4}{\omega_2} \tag{6.10}$$

We can derive this ratio for any linkage by constructing a **pair of effective links** as shown in Figure 6-10a. The definition of **effective link pairs** is *two lines, mutually parallel, drawn through the fixed pivots and intersecting the coupler extended*. These are shown as O_2A' and O_4B' in Figure 6-10a. Note that there is an infinity of possible effective link pairs. They must be parallel to one another but may make any angle with link 3. In the figure they are shown perpendicular to link 3 for convenience in the derivation to follow. The angle between links 2 and 3 is shown as v. The transmission angle between links 3 and 4 is μ . We will now derive an expression for the angular velocity ratio using these effective links, the actual link lengths, and angles v and μ .

From geometry:

$$O_2 A' = (O_2 A) \sin \nu$$
 $O_4 B' = (O_4 B) \sin \mu$ (6.11a)

From equation 6.7

$$V_{A'} = (O_2 A')\omega_2$$
 (6.11b)

The component of velocity $v_{A'}$ lies along the link *AB*. Just as with a two-force member in which a force applied at one end transmits only its component that lies along the link to the other end, this velocity component can be transmitted along the link to point *B*. This is sometimes called the **principle of transmissibility**. We can then equate these components at either end of the link.

$$v_{A'} = v_{B'} \tag{6.11c}$$

Then:

$$O_2 A' \omega_2 = O_4 B' \omega_4 \tag{6.11d}$$

rearranging:

$$\frac{\omega_4}{\omega_2} = \frac{O_2 A}{O_4 B'} \tag{6.11e}$$

and substituting:

$$\frac{\omega_4}{\omega_2} = \frac{O_2 A \sin \nu}{O_4 B \sin \mu} = m_V \tag{6.11f}$$

Note in equation 6.11f that as angle v goes through zero, the angular velocity ratio will be zero regardless of the values of ω_2 or the link lengths, and thus ω_4 will be zero. When angle v is zero, links 2 and 3 will be colinear and thus be in their toggle positions. We learned in Section 3.3 (p. 80) that the limiting positions of link 4 are defined by these toggle conditions. We should expect that the velocity of link 4 will be zero when it has come to the end of its travel. An even more interesting situation obtains if we allow angle μ to go to zero. Equation 6.11f shows that ω_4 will go to infinity when $\mu = 0$, regardless of the values of ω_2 or the link lengths. We clearly cannot allow μ to reach zero. In fact, we learned in Section 3.3 that we should keep this transmission angle μ above about 40 degrees to maintain good quality of motion and force transmission.*

Figure 6-10b shows the same linkage as in Figure 6-10a, but the effective links have now been drawn so that they are not only parallel but are colinear, and thus lie on top of one another. Both intersect the extended coupler at the same point, which is instant center $I_{2,4}$. So, A' and B' of Figure 6-10a are now coincident at $I_{2,4}$. This allows us to write an equation for the **angular velocity ratio** in terms of the distances from the fixed pivots to instant center $I_{2,4}$.

$$m_V = \frac{\omega_4}{\omega_2} = \frac{O_2 I_{2,4}}{O_4 I_{2,4}}$$
(6.11g)

Thus, the instant center $I_{2,4}$ can be used to determine the **angular velocity ratio**.

* This limitation on transmission angle is only critical if the output load is applied to a link that is pivoted to ground (to link 4 in the case of a fourbar linkage). If the load is applied to a floating link (coupler), then other measures of the quality of force transmission than the transmission angle are more appropriate, as discussed in Chapter 12.



FIGURE 6-10

Effective links and the angular velocity ratio

Mechanical Advantage

The power P in a mechanical system can be defined as the dot or scalar product of the force vector \mathbf{F} and the velocity vector \mathbf{V} at any point :

$$P = \mathbf{F} \cdot \mathbf{V} = F_x V_x + F_y V_y \tag{6.12a}$$

For a rotating system, power P becomes the product of torque T and angular velocity ω which, in two dimensions, have the same (z) direction:

$$P = T\omega \tag{6.12b}$$

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The power flows through a passive system and:

$$P_{in} = P_{out} + losses \tag{6.12c}$$

Mechanical efficiency can be defined as:

$$\varepsilon = \frac{P_{out}}{P_{in}} \tag{6.12d}$$

Linkage systems can be very efficient if they are well made with low friction bearings on all pivots. Losses are often less than 10%. For simplicity in the following analysis we will assume that the losses are zero (i.e., a conservative system). Then, letting T_{in} and ω_{in} represent input torque and angular velocity, and T_{out} and ω_{out} represent output torque and angular velocity, then:

$$P_{in} = T_{in}\omega_{in}$$

$$P_{out} = T_{out}\omega_{out}$$
(6.12e)

and :

$$P_{out} = P_{in}$$

$$T_{out}\omega_{out} = T_{in}\omega_{in}$$

$$\frac{T_{out}}{T_{in}} = \frac{\omega_{in}}{\omega_{out}}$$
(6.12f)

Note that the **torque ratio** $(m_T = T_{out}/T_{in})$ is the inverse of the angular velocity ratio.

Mechanical advantage (m_A) can be defined as:

$$m_A = \frac{F_{out}}{F_{in}} \tag{6.13a}$$

Assuming that the input and output forces are applied at some radii r_{in} and r_{out} , perpendicular to their respective force vectors,

$$F_{out} = \frac{T_{out}}{r_{out}}$$

$$F_{in} = \frac{T_{in}}{r_{in}}$$
(6.13b)

substituting equations 6.13b in 6.13a gives an expression in terms of torque.

$$m_A = \left(\frac{T_{out}}{T_{in}}\right) \left(\frac{r_{in}}{r_{out}}\right)$$
(6.13c)

Substituting equation 6.12f in 6.13c gives

$$m_A = \left(\frac{\omega_{in}}{\omega_{out}}\right) \left(\frac{r_{in}}{r_{out}}\right)$$
(6.13d)

and substituting equation 6.11f gives



"Rock-crusher" toggle mechanism

$$m_A = \left(\frac{O_4 B \sin \mu}{O_2 A \sin \nu}\right) \left(\frac{r_{in}}{r_{out}}\right)$$
(6.13e)

See Figure 6-11 and compare equation 6.13e to equation 6.1lf and its discussion under **angular velocity ratio** above. Equation 6.13e shows that for any choice of *rin* and *rout*, the mechanical advantage responds to changes in angles v and /.1 in opposite fashion to that of the angular velocity ratio. If the transmission angle /.1 goes to zero (which we don't want it to do), the mechanical advantage also goes to zero regardless of the amount of input force or torque applied. But, when angle v goes to zero (which it can and does, twice per cycle in a Grashof linkage), the mechanical advantage becomes infinite! This is the principle of a rock-crusher mechanism as shown in Figure 6-11. A quite moderate force applied to link 2 can generate a huge force on link 4 to crush the rock. Of course, we cannot expect to achieve the theoretical output of infinite force or torque magnitude, as the strengths of the links and joints will limit the maximum forces and torques obtainable. Another common example of a linkage which takes advantage of this theoretically infinite mechanical advantage at the toggle position is a ViseGrip locking pliers (see Figure P6-21, p. 296).

These two ratios, **angular velocity ratio** and **mechanical advantage**, provide useful, dimensionless **indices of merit** by which we can judge the relative quality of various linkage designs which may be proposed as solutions.

Using Instant Centers in linkage Design

In addition to providing a quick numerical velocity analysis, instant center analysis more importantly gives the designer a remarkable overview of the linkage's global behavior. It is quite difficult to mentally visualize the complex motion of a "floating" coupler link even in a simple fourbar linkage, unless you build a model or run a computer simulation. Because this complex coupler motion in fact reduces to an instantaneous pure rotation about the instant center h, 3, finding that center allows the designer to visualize the motion of the coupler as a pure rotation. One can literally *see* the motion and the directions of velocities of any points of interest by relating them to the instant center. It is only necessary to draw the linkage in a few positions of interest, showing the instant center locations for each position.

Figure 6-12 shows a practical example of how this visual, qualitative analysis technique could be applied to the design of an automobile rear suspension system. Most automobile suspension mechanisms are either fourbar linkages or fourbar slider-cranks, with the wheel assembly carried on the coupler (as was also shown in Figure 3-19, p. 108). Figure 6-l2a shows a rear suspension design from a domestic car of 1970's vintage which was later redesigned because of a disturbing tendency to "bump steer," i.e., turn the rear axle when hitting a bump on one side of the car. The figure is a view looking from the center of the car outward, showing the fourbar linkage which controls the up and down motion of one side of the rear axle and one wheel. Links 2 and 4 are pivoted to the frame of the car which is link 1. The wheel and axle assembly is rigidly attached to the coupler, link 3. Thus the wheel assembly has complex motion in the vertical plane. Ideally, one would like the wheel to move up and down in a straight vertical line when hitting a bump. Figure 6-12b shows the motion of the wheel and the new instant center (I1,3) location for the situation when one wheel has hit a bump. The velocity vector for the center of the wheel in each position is drawn perpendicular to its radius





from *Ir,3.* You can see that the wheel center has a significant horizontal component of motion as it moves up over the bump. This horizontal component causes the wheel center on that side of the car to move forward while it moves upward, thus turning the axle (about a vertical axis) and steering the car with the rear wheels in the same way that you steer a toy wagon. Viewing the path of the instant center over some range of motion gives a clear picture of the behavior of the coupler link. The undesirable behavior of this suspension linkage system could have been predicted from this simple instant center analysis before ever building the mechanism.

Another practical example of the effective use of instant centers in linkage design is shown in Figure 6-13, which is an optical adjusting mechanism used to position a mirror and allow a small amount of rotational adjustment. [1] A more detailed account of this design case study [2] is provided in Chapter 18. The designer, K. Towfigh, recognized that Ir,3 at point E is an instantaneous "fixed pivot" and will allow very small pure rotations about that point with very small translational error. He then designed a one-piece, plastic fourbar linkage whose "pin joints" are thin webs of plastic which flex to allow slight rotation. This is termed a compliant linkage, one that uses elastic deformations of the links as hinges instead of pin joints. He then placed the mirror on the coupler at 11,3. Even the fixed link 1 is the same piece as the "movable links" and has a small set screw to provide the adjustment. A simple and elegant design.

6.5 CENTRODES

Figure 6-14 illustrates the fact that the successive positions of an instant center (or centro) form a path of their own. *This path, or locus, of the instant center is called the* centrode. Since there are two links needed to create an instant center, there will be two centrodes associated with anyone instant center. These are formed by projecting the path of the instant center first on one link and then on the other. Figure 6-14a shows the locus of instant center Ir,3 as projected onto link 1. Because link I is stationary, or fixed, this is called the fixed centrode. By temporarily inverting the mechanism and fixing link 3



The Final Product of Keivan Towfigh

FIGURE 6-13

An optical adjustment linkage (Reproduced from reference (2) with permission)



FIGURE 6-14

Open-loop fixed and moving centrodes (or polodes) of a fourbar linkage

as the ground link, as shown in Figure 6-14b, we can move link 1 as the coupler and project the locus of 11,3 onto link 3. In the original linkage, link 3 was the moving coupler, so this is called the moving centrode. Figure 6-14c shows the original linkage with both fixed and moving centrodes superposed.

The definition of the instant center says that both links have the same velocity at that point, at that instant. Link 1 has zero velocity everywhere, as does the fixed centrode. So, as the linkage moves, the moving centrode must roll against the fixed centrode without slipping. If you cut the fixed and moving centrodes out of metal, as shown in Figure 6-14d, and roll the moving centrode (which is link 3) against the fixed centrode (which is link 1), the complex motion of link 3 will be identical to that of the original linkage. *All of the coupler curves of points on link 3 will have the same path shapes as in the originallinkage.* We now have, in effect, a "linkless" fourbar linkage, really one composed of two bodies which have these centrode shapes rolling against one another. Links 2 and 4 have been eliminated. Note that the example shown in Figure 6-14 is a non-Grashof fourbar. The lengths of its centrodes are limited by the double-rocker toggle positions.

All instant centers of a linkage will have centrodes. If the links are directly connected by a joint, such as lz,3, 13,4, h,2, and 11,4, their fixed and moving centrodes will degenerate to a point at that location on each link. The most interesting centrodes are those involving links not directly connected to one another such as 11,3 and h,4. If we look at the double-crank linkage in Figure 6-15a in which links 2 and 4 both revolve fully, we see that the centrodes of 11,3 form closed curves. The motion of link 3 with respect to link 1 could be duplicated by causing these two centrodes to roll against one another without slipping. Note that there are two loops to the moving centrode. Both must roll on the single-loop fixed centrode to complete the motion of the equivalent double-crank linkage.

We have so far dealt largely with the instant center 11,3. Instant center 12,4 involves two links which are each in pure rotation and not directly connected to one another. If we use a special-case Grashoflinkage with the links crossed (sometimes called an antiparallelogram linkage), the centrodes of 12,4 become ellipses as shown in Figure 6-15b. To guarantee no slip, it will probably be necessary to put meshing teeth on each centrode. We then will have a pair of elliptical, noncircular gears, or gearset, which gives the same output motion as the original double-crank linkage and will have the same variations in the angular velocity ratio and mechanical advantage as the linkage had. Thus we can see that gearsets are also just fourbar linkages in disguise. Noncircular gears find much use in machinery, such as printing presses, where rollers must be speeded and slowed with some pattern during each cycle or revolution. More complicated shapes of noncircular gears are analogous to cams and followers in that the equivalent fourbar linkage must have variable-length links. Circular gears are just a special case of noncircular gears which give a constant angular velocity ratio and are widely used in all machines. Gears and gearsets will be dealt with in more detail in Chapter 10.

In general, centrodes of crank-rockers and double- or triple-rockers will be open curves with asymptotes. Centrodes of double-crank linkages will be closed curves. Program FOURBARwill calculate and draw the fixed and moving centrodes for any linkage input to it. Input the datafiles F06-14.4br, F06-15aAbr, and F06-15bAbr into program FOURBARto see the centrodes of these linkage drawn as the linkages rotate.



Closed-loop fixed and moving centrodes

A "linkless" linkage

A common example of a mechanism made of centrodes is shown in Figure 6-16a. You have probably rocked in a Boston or Hitchcock rocking chair and experienced the soothing motions that it delivers to your body. You may have also rocked in a *platfonn* rocker as shown in Figure 6-16b and noticed that its motion did not feel as soothing.

There are good kinematic reasons for the difference. The platform rocker has a fixed pin joint between the seat and the base (floor). Thus all parts of your body are in pure rotation along concentric arcs. You are in effect riding on the rocker of a linkage.

The Boston rocker has a shaped (curved) base, or "runners," which rolls against the floor. These runners are usually not circular arcs. They have a higher-order curve contour. They are, in fact, moving centrodes. The floor is the fixed centrode. When one is rolled against the other, the chair and its occupant experience coupler curve motion. Every part of your body travels along a different sixth-order coupler curve which provides smooth accelerations and velocities and feels better than the cruder second-order (circular) motion of the platform rocker. Our ancestors, who carved these rocking chairs,

probably had never heard of fourbar linkages and centrodes, but they knew intuitively how to create comfortable motions.

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Another example of a centrode which you probably use frequently is the path of the tire on your car or bicycle. As your tire rolls against the road without slipping, the road becomes a fixed centrode and the circumference of the tire is the moving centrode. The tire is, in effect, the coupler of a linkless fourbar linkage. All points on the contact surface of the tire move along cycloidal coupler curves and pass through a cusp of zero velocity when they reach the fixed centrode at the road surface as shown in Figure 6-17 a. All other points on the tire and wheel assembly travel along coupler curves which do not have cusps. This last fact is a clue to a means to identify coupler points which will have cusps in their coupler curve. If a coupler point is chosen to be on the moving centrode at one extreme of its path motion (i.e., at one of the positions of h,3), then it will have a cusp in its coupler curve. Figure 6-17b shows a coupler curve of such a point, drawn with program FOURBAR. The right end of the coupler path touches the moving centrode and as a result has a cusp at that point. So, if you desire a cusp in your coupler motion, many are available. Simply choose a coupler point on the moving centrode of link 3. Read the diskfile F06-17bAbr into program FOURBARto animate that linkage with its coupler curve or centrodes. Note in Figure 6-14 (p. 264) that choosing any location of instant center 11,3 on the coupler as the coupler point will provide a cusp at that point.

6.6 VELOCITY OF SLIP

When there is a sliding joint between two links and neither one is the ground link, the velocity analysis is more complicated. Figure 6-18 shows an inversion of the fourbar slider-crank mechanism in which the sliding joint is floating, i.e., not grounded. To solve for the velocity at the sliding joint A, we have to recognize that there is more than one point A at that joint. There is a point A as part of link 2 (Az), a point A as part of link 3 (A3), and a point A as part of link 4 (A_{4}). This is a CASE2 situation in which we have at least two points belonging to different links but occupying the same location at a given instant. Thus, the relative velocity equation 6.6 (p. 243) will apply. We can usually solve for the velocity of at least one of these points directly from the known input information using equation 6.7 (p. 244). It and equation 6.6 are all that are needed to solve for everything else. In this example link 2 is the driver, and 8z and 00z are given for the "freeze frame" position shown. We wish to solve for 004, the angular velocity of link 4, and also for the velocity of slip at the joint labeled A.

In Figure 6-18 the axis of slip is shown to be tangent to the slider motion and is the line along which all sliding occurs between links 3 and 4. The axis of transmission is defined to be perpendicular to the axis of slip and pass through the slider joint at A. This axis of transmission is the only line along which we can transmit motion or force across the slider joint, except for friction. We will assume friction to be negligible in this example. Any force or velocity vector applied to point A can be resolved into two components along these two axes which provide a translating and rotating, local coordinate system for analysis at the joint. The component along the axis of transmission will do useful work at the joint. But, the component along the axis of slip does no work, except friction work.



(a) Boston rocker



(b) Platform rocker

FIGURE 6-16

Some rocking chairs use centrodes of a fourbar linkage



(a) Cycloidal motion of a circular, moving centrode rolling on a straight, fixed centrode



(b) Coupler curve cusps exist only on the moving centrode

FIGURE 6-17

Examples of centrodes

EDEXAMPLE 6-5

Graphical Velocity Analysis at a Sliding Joint.

Problem: Given θ_2 , θ_3 , θ_4 , ω_2 , find ω_3 , ω_4 , V_A , by graphical methods.

Solution: (see Figure 6-18)

1 Start at the end of the linkage for which you have the most information. Calculate the magnitude of the velocity of **point A as part of link 2** (A_2) using scalar equation 6.7 (p. 244).

$$v_{A_2} = (AO_2)\omega_2 \tag{a}$$

- 2 Draw the velocity vector \mathbf{V}_{A_2} with its length equal to its magnitude v_{A_2} at some convenient scale and with its root at point A and its direction perpendicular to the radius AO_2 . Its sense is the same as that of ω_2 as is shown in Figure 6-18.
- 3 Draw the axis of slip and axis of transmission through point A.
- 4 Project V_{A_2} onto the axis of slip and onto the axis of transmission to create the components $V_{A_{2slip}}$ and V_{trans} of V_{A_2} on the axes of slip and transmission, respectively. Note that the **transmission component** is shared by all true velocity vectors at this point, as it is the only component which can transmit across the joint.
- 5 Note that link 3 is pin-jointed to link 2, so $V_{A_3} = V_{A_2}$.
- 6 Note that the direction of the velocity of point V_{A_4} is predictable since all points on link 4 are pivoting in pure rotation about point O_4 . Draw the line *pp* through point A and perpendicular to the effective link 4, AO_4 . Line *pp* is the direction of velocity V_{A_4} .
- 7 Construct the true magnitude of velocity vector V_{A_4} by extending the projection of the **transmission component** V_{trans} until it intersects line *pp*.
- 8 Project V_{A_4} onto the axis of slip to create the slip component $V_{A_{4slip}}$.
- 9 Write the relative velocity vector equation 6.6 (p. 243) for the **slip components** of point A_2 versus point A_4 .

$$V_{slip_{42}} = V_{A_{4slip}} - V_{A_{2slip}} \tag{b}$$



FIGURE 6-18

Velocity of slip and velocity of transmission (note that the applied ω is negative as shown)

10 The angular velocities of links 3 and 4 are identical because they share the slider joint and must rotate together. They can be calculated from equation 6.7:

$$\omega_4 = \omega_3 = \frac{V_{A_4}}{AO_4} \tag{c}$$

Instant center analysis can also be used to solve sliding joint velocity problems graphically.

EDEXAMPLE 6-6

Graphical Velocity Analysis in the Fourbar Inverted Slider-Crank Mechanism Using Instant Centers.

Problem: Given θ_2 , θ_3 , θ_4 , ω_2 , find ω_3 , ω_4 , \mathbf{V}_A , by graphical methods.

Solution: (see Figure 6-19)

1 Start at the end of the linkage about which you have the most information. Calculate the magnitude of the velocity of point A as part of link 2 (A_2) using scalar equation 6.7 (p. 244).

$$v_{A_2} = (AO_2)\omega_2 \tag{a}$$

- 2 Draw the velocity vector \mathbf{V}_{A_2} with its length equal to its magnitude v_{A_2} at some convenient scale and with its root at point A and its direction perpendicular to the radius AO_2 . Its sense is the same as that of ω_2 as is shown in Figure 6-19. Note that link 3 is pin-jointed to link 2, so $\mathbf{V}_{A_3} = \mathbf{V}_{A_2}$.
- 3 Find the instant centers of the linkage as shown in Figure 6-19.
- 4 Define a point (B) on the slider block for analysis. Draw the **axis of slip** and **axis of transmission** through point B. Note that point B is a multiple point, belonging to both link 3 and link 4, and has different linear velocities in each.
- 5 Project V_{A_2} onto the axis of slip to create the orthogonal component $V_{A_{3slip}}$ along link 3. Translate this slip component along link 3 and place it at point *B*. Rename it $V_{B_{3slip}}$.
- 6 The direction of the true velocity of point *B* as part of link 3 (V_{B_3}) is along a line perpendicular to the radius from $I_{1,3}$ to *B*. Construct a perpendicular to $V_{B_{3slin}}$ at its tip and create V_{B_3} .
- 7 Project V_{B_3} onto the axis of transmission to create the component V_{trans} . Note that the **transmission component** is shared by all true velocity vectors at this point, as it is the only component which can transmit across the joint.
- 8 Note that the direction of the velocity of point V_{B_4} is predictable since all points on link 4 are pivoting in pure rotation about point O_4 . Construct a line in the direction of V_{B_4} perpendicular to the effective link 4. Construct the true magnitude of velocity vector V_{B_4} by extending the projection of the transmission component V_{trans} until it intersects the line of V_{B_4} .
- 9 Project V_{B_4} onto the axis of slip to create the slip component $V_{B_{4slip}}$.





Graphical velocity analysis of an inverted slider-crank linkage

10 The total slip velocity at *B* is the difference between the two slip components. Write the relative velocity vector equation 6.6 (p. 243) for the slip components of point B_3 versus point B_4 .

$$V_{slip_{34}} = V_{B_{3slip}} - V_{B_{4slip}} \tag{b}$$

11 The angular velocities of links 3 and 4 are identical because they share the slider joint and must rotate together. They can be calculated from equation 6.7:

$$\omega_4 = \omega_3 = \frac{V_{B_4}}{AO_4} \tag{c}$$

The above examples show how a sliding joint linkage can be solved graphically for velocities at one position. In the next section, we will develop the general solution using algebraic equations to solve the same type of problem.

6.7 ANALYTICAL SOLUTIONS FOR VELOCITY ANALYSIS

The Fourbar Pin-Jointed Linkage

The position equations for the fourbar pin-jointed linkage were derived in Section 4.5 (p. 152). The linkage was shown in Figure 4-7 (p. 155) and is shown again in Figure

6-20 on which we also show an input angular velocity ω_2 applied to link 2. This ω_2 can be a time-varying input velocity. The vector loop equation is shown in equations 4.5a and 4.5c, repeated here for your convenience.

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.5a}$$

As before, we substitute the complex number notation for the vectors, denoting their scalar lengths as *a*, *b*, *c*, *d* as shown in Figure 6-20a.

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0$$
(4.5c)

To get an expression for velocity, differentiate equation 4.5c with respect to time.

$$jae^{j\theta_2} \frac{d\theta_2}{dt} + jbe^{j\theta_3} \frac{d\theta_3}{dt} - jce^{j\theta_4} \frac{d\theta_4}{dt} = 0$$
(6.14a)

But,

$$\frac{d\theta_2}{dt} = \omega_2; \qquad \qquad \frac{d\theta_3}{dt} = \omega_3; \qquad \qquad \frac{d\theta_4}{dt} = \omega_4 \tag{6.14b}$$

and:

$$ja\omega_2 e^{j\theta_2} + jb\omega_3 e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0$$
(6.14c)

Note that the θ_1 term has dropped out because that angle is a constant, and thus its derivative is zero. Note also that equation 6.14 is, in fact the **relative velocity** or **velocity difference equation**.

$$\mathbf{V}_A + \mathbf{V}_{BA} - \mathbf{V}_B = 0 \tag{6.15a}$$

where:

$$\mathbf{V}_{A} = ja \omega_{2} e^{j\theta_{2}}$$

$$\mathbf{V}_{BA} = jb \omega_{3} e^{j\theta_{3}}$$

$$\mathbf{V}_{B} = jc \omega_{4} e^{j\theta_{4}}$$
(6.15b)

Please compare equations 6.15 to equations 6.3, 6.5, and 6.6 (pp. 242 and 243). This equation is solved graphically in the vector diagram of Figure 6-20b.

We now need to solve equation 6.14 for ω_3 and ω_4 , knowing the input velocity ω_2 , the link lengths, and all link angles. Thus the position analysis derived in Section 4.5 must be done first to determine the link angles before this velocity analysis can be completed. We wish to solve equation 6.14 to get expressions in this form:

$$\omega_3 = f(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2) \qquad \omega_4 = g(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2) \tag{6.16}$$

The strategy of solution will be the same as was done for the position analysis. First, substitute the Euler identity from equation 4.4a (p. 155) in each term of equation 6.14c:

$$ja\omega_{2}(\cos\theta_{2} + j\sin\theta_{2}) + jb\omega_{3}(\cos\theta_{3} + j\sin\theta_{3}) -jc\omega_{4}(\cos\theta_{4} + j\sin\theta_{4}) = 0$$
(6.17a)

Multiply through by the operator *j*:







$$a\omega_{2}(j\cos\theta_{2}+j^{2}\sin\theta_{2})+b\omega_{3}(j\cos\theta_{3}+j^{2}\sin\theta_{3})$$
$$-c\omega_{4}(j\cos\theta_{4}+j^{2}\sin\theta_{4})=0$$
(6.17b)

The cosine terms have become the imaginary, or y-directed terms, and because $j^2 = -1$, the sine terms have become real or x-directed.

$$a\omega_{2}(-\sin\theta_{2} + j\cos\theta_{2}) + b\omega_{3}(-\sin\theta_{3} + j\cos\theta_{3})$$
$$-c\omega_{4}(-\sin\theta_{4} + j\cos\theta_{4}) = 0$$
(6.17c)

We can now separate this vector equation into its two components by collecting all real and all imaginary terms separately:

real part (x component):

$$-a\omega_2\sin\theta_2 - b\omega_3\sin\theta_3 + c\omega_4\sin\theta_4 = 0 \tag{6.17d}$$

imaginary part (y component):

$$a\omega_2\cos\theta_2 + b\omega_3\cos\theta_3 - c\omega_4\cos\theta_4 = 0 \tag{6.17e}$$

Note that the j's have cancelled in equation 6.17e. We can solve these two equations, 6.17d and 6.17e, simultaneously by direct substitution to get:

$$\omega_3 = \frac{a\omega_2}{b} \frac{\sin(\theta_4 - \theta_2)}{\sin(\theta_3 - \theta_4)}$$
(6.18a)

$$\omega_4 = \frac{a\omega_2}{c} \frac{\sin(\theta_2 - \theta_3)}{\sin(\theta_4 - \theta_3)}$$
(6.18b)

Once we have solved for ω_3 and ω_4 , we can then solve for the linear velocities by substituting the Euler identity into equations 6.15,

$$\mathbf{V}_A = ja\omega_2(\cos\theta_2 + j\sin\theta_2) = a\omega_2(-\sin\theta_2 + j\cos\theta_2)$$
(6.19a)

$$\mathbf{V}_{BA} = jb\,\omega_3\big(\cos\theta_3 + j\sin\theta_3\big) = b\,\omega_3\big(-\sin\theta_3 + j\cos\theta_3\big) \tag{6.19b}$$

$$\mathbf{V}_B = jc\,\omega_4(\cos\theta_4 + j\sin\theta_4) = c\,\omega_4(-\sin\theta_4 + j\cos\theta_4) \tag{6.19c}$$

where the real and imaginary terms are the *x* and *y* components, respectively. Equations 6.18 and 6.19 provide a complete solution for the angular velocities of the links and the linear velocities of the joints in the pin-jointed fourbar linkage. Note that there are also two solutions to this velocity problem, corresponding to the open and crossed branches of the linkage. They are found by the substitution of the open or crossed branch values of θ_3 and θ_4 obtained from equations 4.10 (p. 158) and 4.13 (p. 159) into equations 6.18 and 6.19. Figure 6-20a (p. 273) shows the open branch.

The Fourbar Slider-Crank

The position equations for the fourbar offset slider-crank linkage (inversion #1) were derived in Section 4.6 (p. 159). The linkage was shown in Figure 4-9 (p. 160) and is shown again in Figure 6-21a on which we also show an input angular velocity ω_2 applied to link 2. This ω_2 can be a time-varying input velocity. The vector loop equation 4.14 is repeated here for your convenience.

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.14a}$$

$$ae^{j\theta_2} - be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0$$
(4.14b)

Differentiate equation 4.14b with respect to time noting that a, b, c, θ_1 , and θ_4 are constant but the length of link d varies with time in this inversion.

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{d} = 0 \tag{6.20a}$$

(6.20b)

The term $d \, dot$ is the linear velocity of the slider block. Equation 6.20a is the velocity difference equation 6.5 (p. 243) and can be written in that form.

 $\mathbf{V}_A - \mathbf{V}_{AB} - \mathbf{V}_B = 0$

or:

but :

then:

 $\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA}$

 $\mathbf{V}_{AB} = -\mathbf{V}_{BA}$

 $\mathbf{V}_A = \mathbf{V}_B + \mathbf{V}_{AB}$

Equation 6.20 is identical in form to equations 6.5 and 6.15a. Note that because we arranged the position vector \mathbf{R}_3 in Figures 4-9 and 6-21 with its root at point *B*, directed from *B* to *A*, its derivative represents the velocity difference of point *A* with respect to point *B*, the opposite of that in the previous fourbar example. Compare this also to equa-



FIGURE 6-21

Position vector loop for a fourbar slider-crank linkage showing velocity vectors for a negative (cw) ω_2

tion 6.15b noting that its vector \mathbf{R}_3 is directed from A to B. Figure 6-21b shows the vector diagram of the graphical solution to equation 6.20b.

Substitute the Euler equivalent, equation 4.4a (p. 155), in equation 6.20a,

$$ja\omega_2(\cos\theta_2 + j\sin\theta_2) - jb\omega_3(\cos\theta_3 + j\sin\theta_3) - d = 0$$
(6.21a)

simplify,

$$a\omega_2(-\sin\theta_2 + j\cos\theta_2) - b\omega_3(-\sin\theta_3 + j\cos\theta_3) - \dot{d} = 0$$
(6.21b)

and separate into real and imaginary components.

real part (x component):

$$-a\omega_2\sin\theta_2 + b\omega_3\sin\theta_3 - d = 0 \tag{6.21c}$$

imaginary part (y component):

$$a\omega_2\cos\theta_2 - b\omega_3\cos\theta_3 = 0 \tag{6.21d}$$

These are two simultaneous equations in the two unknowns, $d \, dot$ and ω_3 . Equation 6.21d can be solved for ω_3 and substituted into 6.21c to find $d \, dot$.

$$\omega_3 = \frac{a}{b} \frac{\cos\theta_2}{\cos\theta_3} \omega_2 \tag{6.22a}$$

$$d = -a\omega_2\sin\theta_2 + b\omega_3\sin\theta_3 \tag{6.22b}$$

The absolute velocity of point A and the velocity difference of point A versus point B are found from equation 6.20:

$$\mathbf{V}_A = a\omega_2 \left(-\sin\theta_2 + j\cos\theta_2\right) \tag{6.23a}$$

$$\mathbf{V}_{AB} = b\,\omega_3 \left(-\sin\theta_3 + j\cos\theta_3\right) \tag{6.23b}$$

$$\mathbf{V}_{BA} = -\mathbf{V}_{AB} \tag{6.23c}$$

The Fourbar Inverted Slider-Crank

The position equations for the fourbar inverted slider-crank linkage were derived in Section 4.7 (p. 161). The linkage was shown in Figure 4-10 (p. 162) and is shown again in Figure 6-22 on which we also show an input angular velocity ω_2 applied to link 2. This ω_2 can vary with time. The vector loop equations 4.14 shown on p. 274 are valid for this linkage as well.

All slider linkages will have at least one link whose effective length between joints varies as the linkage moves. In this inversion the length of link 3 between points A and B, designated as b, will change as it passes through the slider block on link 4. To get an expression for velocity, differentiate equation 4.14b with respect to time noting that a, c, d, and θ_1 are constant and b varies with time.

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{b}e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0$$
(6.24)

The value of *db/dt* will be one of the variables to be solved for in this case and is the *b dot* term in the equation. Another variable will be ω_4 , the angular velocity of link 4. Note, however, that we also have an unknown in ω_3 , the angular velocity of link 3. This is a total of three unknowns. Equation 6.24 can only be solved for two unknowns. Thus we require another equation to solve the system. There is a fixed relationship between angles θ_3 and θ_4 , shown as γ in Figure 6-22 and defined in equation 4.18, repeated here:

 θ_3

ω

$$=\theta_4 \pm \gamma \tag{4.18}$$

Differentiate it with respect to time to obtain:

$$_{3} = \omega_{4} \tag{6.25}$$

We wish to solve equation 6.24 to get expressions in this form:

$$\omega_3 = \omega_4 = f(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2)$$

$$\frac{db}{dt} = \dot{b} = g(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2)$$
(6.26)

Substitution of the Euler identity (equation 4.4a, p. 155) into equation 6.24 yields:

$$ja\omega_{2}(\cos\theta_{2} + j\sin\theta_{2}) - jb\omega_{3}(\cos\theta_{3} + j\sin\theta_{3}) -\dot{b}(\cos\theta_{3} + j\sin\theta_{3}) - jc\omega_{4}(\cos\theta_{4} + j\sin\theta_{4}) = 0$$
(6.27a)

Multiply by the operator *j* and substitute ω_4 for ω_3 from equation 6.25:

$$a\omega_{2}(-\sin\theta_{2} + j\cos\theta_{2}) - b\omega_{4}(-\sin\theta_{3} + j\cos\theta_{3}) -b(\cos\theta_{3} + j\sin\theta_{3}) - c\omega_{4}(-\sin\theta_{4} + j\cos\theta_{4}) = 0$$
(6.27b)



FIGURE 6-22

Velocity analysis of inversion #3 of the slider-crank fourbar linkage

We can now separate this vector equation into its two components by collecting all real and all imaginary terms separately:

real part (x component):

$$-a\omega_2\sin\theta_2 + b\omega_4\sin\theta_3 - b\cos\theta_3 + c\omega_4\sin\theta_4 = 0 \tag{6.28a}$$

imaginary part (y component):

$$a\omega_2\cos\theta_2 - b\omega_4\cos\theta_3 - \dot{b}\sin\theta_3 - c\omega_4\cos\theta_4 = 0$$
(6.28b)

Collect terms and rearrange equations 6.28 to isolate one unknown on the left side.

$$b\cos\theta_3 = -a\omega_2\sin\theta_2 + \omega_4(b\sin\theta_3 + c\sin\theta_4)$$
(6.29a)

$$\dot{b}\sin\theta_3 = a\omega_2\cos\theta_2 - \omega_4(b\cos\theta_3 + c\cos\theta_4)$$
(6.29b)

Either equation can be solved for $b \, dot$ and the result substituted in the other. Solving equation 6.29a:

$$\dot{b} = \frac{-a\omega_2\sin\theta_2 + \omega_4(b\sin\theta_3 + c\sin\theta_4)}{\cos\theta_3}$$
(6.30a)

Substitute in equation 6.29b and simplify:

$$\omega_4 = \frac{a\omega_2\cos(\theta_2 - \theta_3)}{b + c\cos(\theta_4 - \theta_3)}$$
(6.30b)

Equation 6.30a provides the **velocity of slip** at point *B*. Equation 6.30b gives the **angular velocity** of link 4. Note that we can substitute $-\gamma = \theta_4 - \theta_3$ from equation 4.18 (for an open linkage) into equation 6.30b to further simplify it. Note that $\cos(-\gamma) = \cos(\gamma)$.

$$\omega_4 = \frac{a\omega_2\cos(\theta_2 - \theta_3)}{b + \cos\gamma} \tag{6.30c}$$

The **velocity of slip** from equation 6.30a is always directed along the **axis of slip** as shown in Figures 6-19 (p. 271) and 6-22 (p. 277). There is also a component orthogonal to the axis of slip called the **velocity of transmission**. This lies along the **axis of transmission** which is the only line along which any useful work can be transmitted across the sliding joint. All energy associated with motion along the slip axis is converted to heat and lost.

The absolute linear velocity of point A is found from equation 6.23a. We can find the absolute velocity of point B on link 4 since ω_4 is now known. From equation 6.15b:

$$\mathbf{V}_{B_4} = jc\,\omega_4 e^{j\Theta_4} = c\,\omega_4 \left(-\sin\theta_4 + j\cos\theta_4\right) \tag{6.31}$$

6.8 VELOCITY ANALYSIS OF THE GEARED FIVEBAR LINKAGE

The position loop equation for the geared fivebar mechanism was derived in Section 4.8 and is repeated here. See Figure P6-4 for notation.

$$ae^{j\theta_{2}} + be^{j\theta_{3}} - ce^{j\theta_{4}} - de^{j\theta_{5}} - fe^{j\theta_{1}} = 0$$
(4.23b)

Differentiate this with respect to time to get an expression for velocity.

$$a\omega_{2}je^{j\theta_{2}} + b\omega_{3}je^{j\theta_{3}} - c\omega_{4}je^{j\theta_{4}} - d\omega_{5}je^{j\theta_{5}} = 0$$
(6.32a)

Substitute the Euler equivalents:

$$a\omega_2 j(\cos\theta_2 + j\sin\theta_2) + b\omega_3 j(\cos\theta_3 + j\sin\theta_3) -c\omega_4 j(\cos\theta_4 + j\sin\theta_4) - d\omega_5 j(\cos\theta_5 + j\sin\theta_5) = 0$$
(6.32b)

Note that the angle θ_5 is defined in terms of θ_2 , the gear ratio λ , and the phase angle ϕ .

$$\theta_5 = \lambda \theta_2 + \phi \tag{4.23c}$$

Differentiate with respect to time:

$$\omega_5 = \lambda \omega_2 \tag{6.32c}$$

Since a complete position analysis must be done before a velocity analysis, we will assume that the values of θ_5 and ω_5 have been found and will leave these equations in terms of θ_5 and ω_5 .

Separating the real and imaginary terms in equation 6.32b:

real: $-a\omega_2\sin\theta_2 - b\omega_3\sin\theta_3 + c\omega_4\sin\theta_4 + d\omega_5\sin\theta_5 = 0$ (6.32d)

imaginary:
$$a\omega_2\cos\theta_2 + b\omega_3\cos\theta_3 - c\omega_4\cos\theta_4 - d\omega_5\cos\theta_5 = 0$$
 (6.32e)

The only two unknowns are ω_3 and ω_4 . Either equation 6.32d or 6.32e can be solved for one unknown and the result substituted in the other. The solution for ω_3 is:

$$\omega_3 = -\frac{2\sin\theta_4 \left[a\omega_2 \sin(\theta_2 - \theta_4) + d\omega_5 \sin(\theta_4 - \theta_5) \right]}{b \left[\cos(\theta_3 - 2\theta_4) - \cos\theta_3 \right]}$$
(6.33a)

The angular velocity ω_4 can be found from equation 6.32d using ω_3 .

$$\omega_4 = \frac{a\omega_2\sin\theta_2 + b\omega_3\sin\theta_3 - d\omega_5\sin\theta_5}{c\sin\theta_4}$$
(6.33b)

With all link angles and angular velocities known, the linear velocities of the pin joints can be found from:

$$\mathbf{V}_A = a\,\omega_2 \left(-\sin\theta_2 + j\cos\theta_2\right) \tag{6.33c}$$

$$\mathbf{V}_{BA} = b\,\omega_3 \big(-\sin\theta_3 + j\cos\theta_3\big) \tag{6.33d}$$

$$\mathbf{V}_{C} = d\,\omega_{5} \left(-\sin\theta_{5} + j\cos\theta_{5}\right) \tag{6.33e}$$

$$\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA} \tag{6.33f}$$

6.9 VELOCITY OF ANY POINT ON A LINKAGE

Once the angular velocities of all the links are found it is easy to define and calculate the velocity of *any point on any link* for any input position of the linkage. Figure 6-23 shows the fourbar linkage with its coupler, link 3, enlarged to contain a coupler point P. The crank and rocker have also been enlarged to show points S and U which might represent the centers of gravity of those links. We want to develop algebraic expressions for the velocities of these (or any) points on the links.

To find the velocity of point *S*, draw the position vector from the fixed pivot O_2 to point *S*. This vector, \mathbf{R}_{SO_2} makes an angle δ_2 with the vector \mathbf{R}_{AO_2} . The angle δ_2 is completely defined by the geometry of link 2 and is constant. The position vector for point *S* is then:

$$\mathbf{R}_{SO_2} = \mathbf{R}_S = se^{j(\theta_2 + \delta_2)} = s\left[\cos(\theta_2 + \delta_2) + j\sin(\theta_2 + \delta_2)\right]$$
(4.25)

Differentiate this position vector to find the velocity of that point.

$$\mathbf{V}_{S} = jse^{j(\theta_{2} + \delta_{2})}\omega_{2} = s\omega_{2}\left[-\sin(\theta_{2} + \delta_{2}) + j\cos(\theta_{2} + \delta_{2})\right]$$
(6.34)

The position of point U on link 4 is found in the same way, using the angle δ_4 which is a constant angular offset within the link. The expression is:

$$\mathbf{R}_{UO_4} = ue^{j(\theta_4 + \delta_4)} = u\left[\cos(\theta_4 + \delta_4) + j\sin(\theta_4 + \delta_4)\right]$$
(4.26)

Differentiate this position vector to find the velocity of that point.

$$\mathbf{V}_{U} = jue^{j(\theta_{4} + \delta_{4})}\boldsymbol{\omega}_{4} = u\boldsymbol{\omega}_{4} \left[-\sin(\theta_{4} + \delta_{4}) + j\cos(\theta_{4} + \delta_{4}) \right]$$
(6.35)

The velocity of point *P* on link 3 can be found from the addition of two velocity vectors, such as V_A and V_{PA} . V_A is already defined from our analysis of the link velocities.



Finding the velocities of points on the links

 \mathbf{V}_{PA} is the velocity difference of point *P* with respect to point *A*. Point *A* is chosen as the reference point because angle θ_3 is defined in a local coordinate system whose origin is at *A*. Position vector \mathbf{R}_{PA} is defined in the same way as \mathbf{R}_S or \mathbf{R}_U using the internal link offset angle δ_3 and the angle of link 3, θ_3 . This was done in equation 4.27.

$$\mathbf{R}_{PA} = pe^{j(\theta_3 + \delta_3)} = p\left[\cos(\theta_3 + \delta_3) + j\sin(\theta_3 + \delta_3)\right]$$
(4.27a)

$$\mathbf{R}_{P} = \mathbf{R}_{A} + \mathbf{R}_{PA} \tag{4.27b}$$

Differentiate this position vector to find the velocity of that point.

$$\mathbf{V}_{PA} = jpe^{j(\theta_3 + \delta_3)}\omega_3 = p\omega_3 \left[-\sin(\theta_3 + \delta_3) + j\cos(\theta_3 + \delta_3)\right]$$
(6.36a)

$$\mathbf{V}_P = \mathbf{V}_A + \mathbf{V}_{PA} \tag{6.36b}$$

Please compare equation 6.36 with equations 6.5 (p. 243) and 6.15 (p. 272). It is, again, the velocity difference equation.

6.10 REFERENCES

- 1 **Towfigh, K.** (1969). "The Fourbar Linkage as an Adjustment Mechanism." *Proc. of Applied Mechanism Conference*, Tulsa, OK, pp. 27-1 to 27-4.
- 2 Wood, G. A. (1977). "Educating for Creativity in Engineering." *Proc. of ASEE 85th Annual Conference*, University of North Dakota, pp. 1-13.