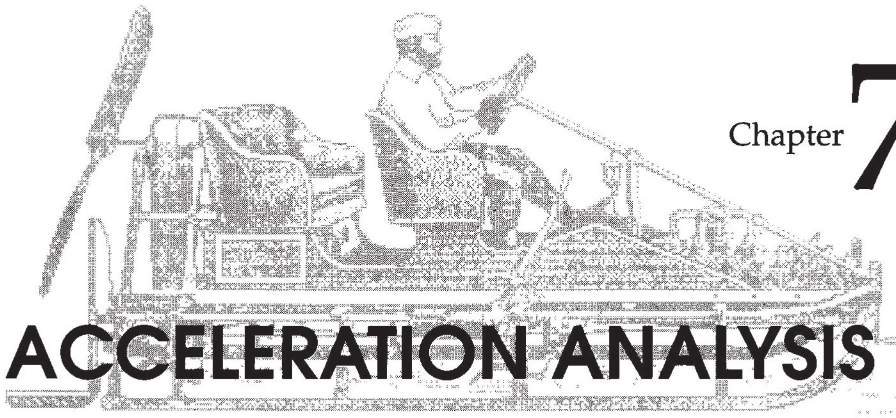


Chapter 7

ACCELERATION ANALYSIS

Take it to warp five, Mr. Sulu
CAPTAIN KIRK

7.0 INTRODUCTION

Once a velocity analysis is done, the next step is to determine the accelerations of all links and points of interest in the mechanism or machine. We need to know the accelerations to calculate the dynamic forces from $\mathbf{F} = m\mathbf{a}$. The dynamic forces will contribute to the stresses in the links and other components. Many methods and approaches exist to find accelerations in mechanisms. We will examine only a few of these methods here. We will first develop a manual graphical method, which is often useful as a check on the more complete and accurate analytical solution. Then we will derive the analytical solution for accelerations in the fourbar and inverted slider-crank linkages as examples of the general vector loop equation solution to acceleration analysis problems.

7.1 DEFINITION OF ACCELERATION

Acceleration is defined as *the rate of change of velocity with respect to time*. Velocity ($\mathbf{V}$, ω) is a vector quantity and so is acceleration. Accelerations can be **angular** or **linear**. **Angular acceleration** will be denoted as α and **linear acceleration** as $\mathbf{A}$.

$$\alpha = \frac{d\omega}{dt}; \qquad \mathbf{A} = \frac{d\mathbf{V}}{dt} \tag{7.1}$$

Figure 7-1 shows a link PA in pure rotation, pivoted at point A in the xy plane. We are interested in the acceleration of point P when the link is subjected to an angular velocity ω and an angular acceleration α, which need not have the same sense. The link's position is defined by the position vector $\mathbf{R}$, and the velocity of point P is $\mathbf{V}_{PA}$. These vectors were defined in equations 6.2 and 6.3 which are repeated here for convenience. (See also Figure 6-1, p. 242.)

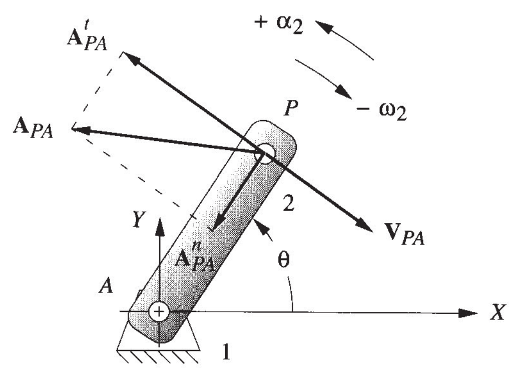


FIGURE 7-1

Acceleration of a link in pure rotation with a positive (CCW) α_2 and a negative (CW) ω_2

$$\mathbf{R}_{PA} = pe^{j\theta} \tag{6.2}$$

$$\mathbf{V}_{PA} = \frac{d\mathbf{R}_{PA}}{dt} = pje^{j\theta}\frac{d\theta}{dt} = p\omega je^{j\theta} \tag{6.3}$$

where p is the scalar length of the vector $\mathbf{R}_{PA}$. We can easily differentiate equation 6.3 to obtain an expression for the acceleration of point P:

$$\begin{aligned}
\mathbf{A}_{PA} &= \frac{d\mathbf{V}_{PA}}{dt} = \frac{d\left(p\omega je^{j\theta}\right)}{dt} \\
\mathbf{A}_{PA} &= jp\left(e^{j\theta}\frac{d\omega}{dt} + \omega je^{j\theta}\frac{d\theta}{dt}\right) \\
\mathbf{A}_{PA} &= p\alpha je^{j\theta} - p\omega^2 e^{j\theta} \\
\mathbf{A}_{PA} &= \mathbf{A}^t_{PA} + \mathbf{A}^n_{PA}
\end{aligned} \tag{7.2}$$

Note that there are two functions of time in equation 6.3, θ and ω. Thus there are two terms in the expression for acceleration, the tangential component of acceleration $\mathbf{A}^t_{PA}$ involving α, and the normal (or centripetal) component $\mathbf{A}^n_{PA}$ involving ω^2. As a result of the differentiation, the tangential component is multiplied by the (constant) complex operator j. This causes a rotation of this acceleration vector through 90 ° with respect to the original position vector. (See also Figure 4-5b, p. 152.) This 90° rotation is nominally positive, or counterclockwise (CCW). However, the tangential component is also multiplied by α, which may be either positive or negative. As a result, the tangential component of acceleration will be **rotated 90°** from the angle θ of the *position vector* **in a direction dictated by the sign of α.** This is just mathematical verification of what you already knew, namely that *tangential acceleration is always in a direction perpendicular to the radius of rotation and is thus tangent to the path of motion* as shown in Figure 7-1. The normal, or centripetal, acceleration component is multiplied by j^2, or -1. This directs *the centripetal component at 180° to the angle θ of the original position vector,* i.e., toward the center (centripetal means *toward the center*). The total acceleration $\mathbf{A}_{PA}$ of point P is the vector sum of the tangential $\mathbf{A}^t_{PA}$ and normal $\mathbf{A}^n_{PA}$ components as shown in Figure 7-1 and equation 7.2.

Substituting the Euler identity (equation 4.4a, p. 155) into equation 7.2 gives us the real and imaginary (or x and y) components of the acceleration vector.

$$\mathbf{A}_{PA} = p\alpha(-\sin\theta + j\cos\theta) - p\omega^2(\cos\theta + j\sin\theta) \tag{7.3}$$

The acceleration $\mathbf{A}_{PA}$ in Figure 7-1 can be referred to as an **absolute acceleration** since it is referenced to A, which is the origin of the global coordinate axes in that system. As such, we could have referred to it as $\mathbf{A}_P$, with the absence of the second subscript implying reference to the global coordinate system.

Figure 7-2a shows a different and slightly more complicated system in which the pivot A is no longer stationary. It has a known linear acceleration $\mathbf{A}_A$ as part of the translating carriage, link 3. If α is unchanged, the acceleration of point P versus A will be the same as before, but $\mathbf{A}_{PA}$ can no longer be considered an absolute acceleration. It is now an **acceleration difference** and **must** carry the second subscript as $\mathbf{A}_{PA}$. The absolute acceleration $\mathbf{A}_P$ must now be found from the **acceleration difference** equation whose graphical solution is shown in Figure 7-2b:

$$\begin{gathered}\mathbf{A}_P = \mathbf{A}_A + \mathbf{A}_{PA} \\ \left(\mathbf{A}^t_P + \mathbf{A}^n_P\right) = \left(\mathbf{A}^t_A + \mathbf{A}^n_A\right) + \left(\mathbf{A}^t_{PA} + \mathbf{A}^n_{PA}\right)\end{gathered} \tag{7.4}$$

Note the similarity of equation 7.4 to the **velocity difference equation** (equation 6.5, p. 243). Note also that the solution for $\mathbf{A}_P$ in equation 7.4 can be found either by adding the resultant vector $\mathbf{A}_{PA}$ or its normal and tangential components, $\mathbf{A}^n_{PA}$ and $\mathbf{A}^t_{PA}$ to the vector $\mathbf{A}_A$ in Figure 7-2b. The vector $\mathbf{A}_A$ has a zero normal component in this example because link 3 is in pure translation.

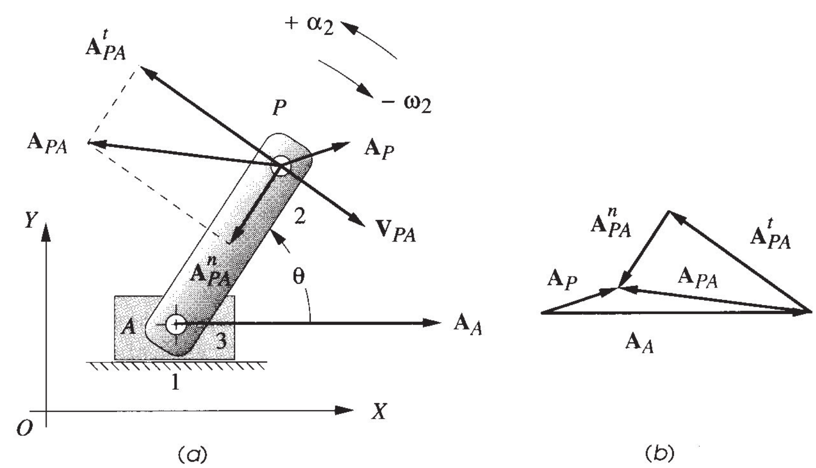


FIGURE 7-2

Acceleration difference in a system with a positive (CCW) α_2 and a negative (CW) ω_2

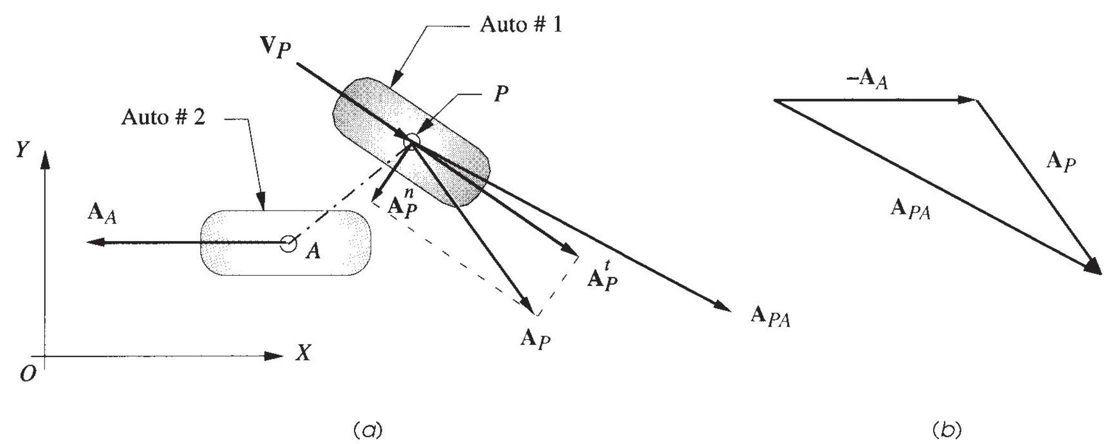


FIGURE 7-3

Relative acceleration

Figure 7-3 shows two independent bodies P and A, which could be two automobiles, moving in the same plane. Auto #1 is turning and accelerating into the path of auto #2, which is decelerating to avoid a crash. If their independent accelerations $\mathbf{A}_P$ and $\mathbf{A}_A$ are known, their **relative acceleration** $\mathbf{A}_{PA}$ can be found from equation 7.4 arranged algebraically as:

$$\mathbf{A}_{PA} = \mathbf{A}_P - \mathbf{A}_A \tag{7.5}$$

The graphical solution to this equation is shown in Figure 7-3b.

As we did for velocity analysis, we give these two cases different names despite the fact that the same equation applies. Repeating the definition from Section 6.1 (p. 241), modified to refer to acceleration:

CASE 1: Two points in the same body => *acceleration difference*

CASE 2: Two points in different bodies => *relative acceleration*

7.2 GRAPHICAL ACCELERATION ANALYSIS

The comments made in regard to graphical velocity analysis in Section 6.2 (p. 244) apply as well to graphical acceleration analysis. Historically, graphical methods were the only practical way to solve these acceleration analysis problems. With some practice, and with proper tools such as a drafting machine or CAD package, one can fairly rapidly solve for the accelerations of particular points in a mechanism for anyone input position by drawing vector diagrams. However, if accelerations for many positions of the mechanism are to be found, each new position requires a completely new set of vector diagrams be drawn. Very little of the work done to solve for the accelerations at position 1 carries over to position 2, etc. This is an even more tedious process than that for graphical velocity analysis because there are more components to draw. Nevertheless, this method still has more than historical value as it can provide a quick check on the results from a computer program solution. Such a check only needs to be done for a few positions to prove the validity of the program.

To solve any acceleration analysis problem graphically, we need only three equations, equation 7.4 and equations 7.6 (which are merely the scalar magnitudes of the terms in equation 7.2, p. 301):

$$\left|\mathbf{A}^t\right| = A^t = r\alpha$$
$$\left|\mathbf{A}^n\right| = A^n = r\omega^2 \tag{7.6}$$

Note that the scalar equations 7.6 define only the **magnitudes** (A^t, A^n) of the components of acceleration of any point in rotation. In a CASE 1 graphical analysis, the **directions** of the vectors due to the centripetal and tangential components of the acceleration difference must be understood from equation 7.2 to be perpendicular to and along the radius of rotation, respectively. Thus, if the center of rotation is known or assumed, the directions of the acceleration difference components due to that rotation are known and their senses will be consistent with the angular velocity ω and angular acceleration α of the body.

Figure 7-4 shows a fourbar linkage in one particular position. We wish to solve for the angular accelerations of links 3 and 4 (α_3, α_4) and the linear accelerations of points *A, B,* and *C* ($\mathbf{A}_A$, $\mathbf{A}_B$, $\mathbf{A}_C$). Point *C* represents any general point of interest such as a coupler point. The solution method is valid for any point on any link. To solve this problem we need to know the *lengths of all the links*, the *angular positions of all the links*, the *angular velocities of all the links,* and the *instantaneous input acceleration of any one driving link or driving point.* Assuming that we have designed this linkage, we will know or can measure the link lengths. We must also first do a **complete position and velocity analysis** to find the link angles θ_3 and θ_4 and angular velocities ω_3 and ω_4 given the input link's position θ_2, input angular velocity ω_2, and input acceleration α_2. This can be done by any of the methods in Chapters 4 and 6. In general we must solve these problems in stages, first for link positions, then for velocities, and finally for accelerations. For the following example, we will assume that a complete position and velocity analysis has been done and that the input is to link 2 with known θ_2, ω_2, and α_2 for this one "freeze-frame" position of the moving linkage.

EXAMPLE 7-1

Graphical Acceleration Analysis for One Position of a Fourbar Linkage.

Problem: Given θ_2, θ_3, θ_4, ω_2, ω_3, ω_4, α_2, find α_3, α_4, $\mathbf{A}_A$, $\mathbf{A}_B$, $\mathbf{A}_P$ by graphical methods.

Solution: (see Figure 7-4)

1 Start at the end of the linkage about which you have the most information. Calculate the magnitudes of the centripetal and tangential components of acceleration of point *A* using scalar equations 7.6.

$$A_A^n = (AO_2)\omega_2^2; \qquad A_A^t = (AO_2)\alpha_2 \tag{a}$$

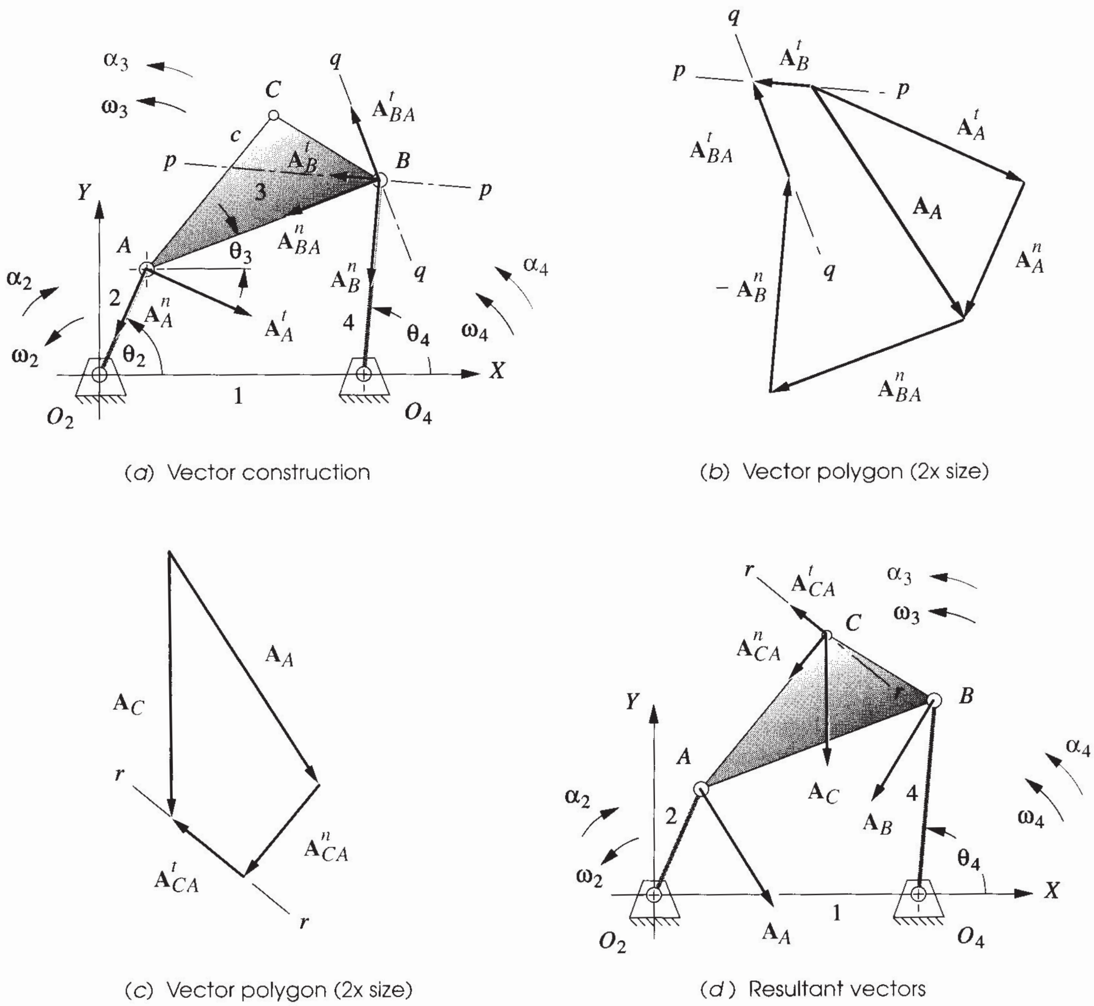


FIGURE 7-4

Graphical solution for acceleration in a pin-jointed linkage with a negative (CW) α_2 and a positive (CCW) ω_2

2 On the linkage diagram, Figure 7-4a, draw the acceleration component vectors $\mathbf{A}_A^n$ and $\mathbf{A}_A^t$ with their lengths equal to their magnitudes at some convenient scale. Place their roots at point A with their directions respectively along and perpendicular to the radius AO_2. The sense of $\mathbf{A}_A^t$ is defined by that of α_2 (according to the right-hand rule), and the sense of $\mathbf{A}_A^n$ is the opposite of that of the position vector $\mathbf{R}_A$ as shown in Figure 7-4a.

3 Move next to a point about which you have some information, such as B on link 4. Note that the directions of the tangential and normal components of acceleration of point B are predictable since this link is in pure rotation about point O_4. Draw the construction line pp through point B perpendicular to BO_4, to represent the direction of $\mathbf{A}_B^t$ as shown in Figure 7-4a.

4 Write the acceleration difference vector equation 7.4 for point B versus point A.

$$\mathbf{A}_B = \mathbf{A}_A + \mathbf{A}_{BA} \tag{b}$$

Substitute the normal and tangential components for each term:

$$\left(\mathbf{A}_B^t + \mathbf{A}_B^n\right) = \left(\mathbf{A}_A^t + \mathbf{A}_A^n\right) + \left(\mathbf{A}_{BA}^t + \mathbf{A}_{BA}^n\right) \tag{c}$$

We will use point A as the reference point to find $\mathbf{A}_B$ because A is in the same link as B and we have already solved for $\mathbf{A}_A^t$ and $\mathbf{A}_A^n$. Any vector equation can be solved for two unknowns. Each term has two parameters, namely magnitude and direction. There are then potentially twelve unknowns in this equation, two per term. We must know ten of them to solve it. We know both the magnitudes and directions of $\mathbf{A}_A^t$ and $\mathbf{A}_A^n$ and the directions of $\mathbf{A}_B^t$ and $\mathbf{A}_B^n$ which are along line pp and line BO_4, respectively. We can also calculate the magnitude of $\mathbf{A}_B^n$ from equation 7.6 since we know ω_4. This provides seven known values. We need to know three more parameters to solve the equation.

5 The term $\mathbf{A}_{BA}$ represents the acceleration difference of B with respect to A. This has two components. The normal component $\mathbf{A}_{BA}^n$ is directed along the line BA because we are using point A as the reference center of rotation for the free vector ω_3, and its magnitude can be calculated from equation 7.6. The direction of $\mathbf{A}_{BA}^t$ must then be perpendicular to the line BA. Draw construction line qq through point B and perpendicular to BA to represent the direction of $\mathbf{A}_{BA}^t$ as shown in Figure 7-4a (p. 305). The calculated magnitude and direction of component $\mathbf{A}_{BA}^n$ and the known direction of $\mathbf{A}_{BA}^t$ provide the needed additional three parameters.

6 Now the vector equation can be solved graphically by drawing a vector diagram as shown in Figure 7-4b. Either drafting tools or a CAD package is necessary for this step. The strategy is to first draw all vectors for which we know both magnitude and direction, being careful to arrange their senses according to equation 7.4 (p. 302).

First draw acceleration vectors $\mathbf{A}_A^t$ and $\mathbf{A}_A^n$ tip to tail, carefully to some scale, maintaining their directions. (They are drawn twice size in the figure.) Note that the sum of these two components is the vector $\mathbf{A}_A$. The equation in step 4 says to add $\mathbf{A}_{BA}$ to $\mathbf{A}_A$. We know $\mathbf{A}_{BA}^n$, so we can draw that component at the end of $\mathbf{A}_A$. We also know $\mathbf{A}_B^n$, but this component is on the left side of equation 7.4, so we must subtract it. Draw the negative (opposite sense) of $\mathbf{A}_B^n$ at the end of $\mathbf{A}_{BA}^n$.

This exhausts our supply of components for which we know both magnitude and direction. Our two remaining knowns are the directions of $\mathbf{A}_B^t$ and $\mathbf{A}_{BA}^t$ which lie along the lines pp and qq, respectively. Draw a line parallel to line qq across the tip of the vector representing *minus* $\mathbf{A}_B^n$. The resultant, or left side of the equation, must close the vector diagram, from the tail of the first vector drawn ($\mathbf{A}_A$) to the tip of the last, so draw a line parallel to pp across the tail of $\mathbf{A}_A$. The intersection of these lines parallel to pp and qq defines the lengths of $\mathbf{A}_B^t$ and $\mathbf{A}_{BA}^t$. The senses of these vectors are determined from reference to equation 7.4. Vector $\mathbf{A}_A$ was added to $\mathbf{A}_{BA}$, so their components must be arranged tip to tail. Vector $\mathbf{A}_B$ is the resultant, so its component $\mathbf{A}_B^t$ must be from the tail of the first to the tip of the last. The resultant vectors are shown in Figure 7-4b and d.

7 The angular accelerations of links 3 and 4 can be calculated from equation 7.6:

$$\alpha_4 = \frac{A_B^t}{BO_4} \qquad \alpha_3 = \frac{A_{BA}^t}{BA} \tag{d}$$

Note that the acceleration difference term $\mathbf{A}^t_{BA}$ represents the rotational component of acceleration of link 3 due to α_3. The rotational acceleration α of any body is a "**free vector**" which has no particular point of application to the body. It exists everywhere on the body.

8 Finally we can solve for $\mathbf{A}_C$ using equation 7.4 again. We select any point in link 3 for which we know the absolute velocity to use as the reference, such as point A.

$$\mathbf{A}_C = \mathbf{A}_A + \mathbf{A}_{CA} \tag{e}$$

In this case, we can calculate the magnitude of $\mathbf{A}^t_{CA}$ from equation 7.6 (p. 304) as we have already found α_3,

$$A^t_{CA} = c\alpha_3 \tag{f}$$

The magnitude of the component $\mathbf{A}^n_{CA}$ can be found from equation 7.6 using ω_3.

$$A^n_{CA} = c\omega_3^2 \tag{g}$$

Since both $\mathbf{A}_A$ and $\mathbf{A}_{CA}$ are known, the vector diagram can be directly drawn as shown in Figure 7-4c. Vector $\mathbf{A}_C$ is the resultant which closes the vector diagram. Figure 7-4d shows the calculated acceleration vectors on the linkage diagram.

The above example contains some interesting and significant principles which deserve further emphasis. Equation 7.4 is repeated here for discussion.

$$\begin{gathered}\mathbf{A}_P = \mathbf{A}_A + \mathbf{A}_{PA} \\ \left(\mathbf{A}^t_P + \mathbf{A}^n_P\right) = \left(\mathbf{A}^t_A + \mathbf{A}^n_A\right) + \left(\mathbf{A}^t_{PA} + \mathbf{A}^n_{PA}\right)\end{gathered} \tag{7.4}$$

This equation represents the *absolute* acceleration of some general point P referenced to the origin of the global coordinate system. The right side defines it as the sum of the absolute acceleration of some other reference point A in the same system and the acceleration difference (or relative acceleration) of point P versus pointA. These terms are then further broken down into their normal (centripetal) and tangential components which have definitions as shown in equation 7.2 (p. 301).

Let us review what was done in Example 7-1 in order to extract the general strategy for solution of this class of problem. We started at the input side of the mechanism, as that is where the driving angular acceleration cx2 was defined. We first looked for a point *(A)* for which the motion was pure rotation. We then solved for the absolute acceleration of that point *(AA)* using equations 7.4 and 7.6 by breaking *AA* into its normal and tangential components. *(Steps* 1 *and 2)*

We then used the point *(A)* just solved for as a reference point to define the translation component in equation 7.4 written for a new point *(B)*. Note that we needed to choose a second point *(B)* which was in the same rigid body as the reference point *(A)* which we had already solved, and about which we could predict some aspect of the new point's *(B's)* acceleration components. **In** this example, we knew the direction of the component A;~ though we did not yet know its magnitude. We could also calculate both magnitude and direction of the centripetal component, $\mathbf{A}_B^n$, since we knew ω_3 and the link length. In general this situation will obtain for any point on a link which is jointed to ground (as is link 4). In this example, we could not have solved for point C until we solved for B, because point C is on a floating link for which we do not yet know the angular acceleration or absolute acceleration direction. *(Steps 3 and 4)*

To solve the equation for the second point (B), we also needed to recognize that the tangential component of the acceleration difference $\mathbf{A}_{BA}^t$ is always directed perpendicular to the line connecting the two related points in the link (B and A in the example). In addition, you will always know the magnitude and direction of the centripetal acceleration components in equation 7.4 ***if* it represents an acceleration difference** (CASE 1) **situation**. *If the two points are in the* ***same rigid*** **body,** *then that acceleration difference centripetal component has a magnitude of $r\omega^2$ and is always directed along the line connecting the two points, pointing toward the reference point as the center* (see Figure 7-2, p. 302). These observations will be true regardless of the two points selected. But, *note this is not true in a* CASE 2 *situation* as shown in Figure 7-3a (p. 303) where the normal component of acceleration of auto #2 is **not** directed along the line connecting points A and P. *(Steps 5 and 6)*

Once we found the absolute acceleration ($\mathbf{A}_B$) of a second point on the same link (CASE 1) we could solve for the angular acceleration of that link. (Note that points A and B are both on link 3 and the acceleration of point O_4 is zero.) Once the angular accelerations of all the links were known, we could solve for the linear acceleration of any point (such as C) in any link using equation 7.4. To do so, we had to understand the concept of angular acceleration as a **free vector**, which means that it exists everywhere on the link at any given instant. It has no particular center. *It has an infinity of potential centers.* The link simply *has an angular acceleration.* It is this property that allows us to solve equation 7.4 for literally **any point** on a rigid body in complex motion **referenced to any other point** on that body. *(Steps 7 and 8)*

7.3 ANALYTICAL SOLUTIONS FOR ACCELERATION ANALYSIS

The Fourbar Pin-Jointed Linkage

The position equations for the fourbar pin-jointed linkage were derived in Section 4.5 (p. 152). The linkage was shown in Figure 4-7 and is shown again in Figure 7-5a on which we also show an input angular acceleration α_2 applied to link 2. This input angular acceleration α_2 may vary with time. The vector loop equation was shown in equations 4.5a and c, repeated here for your convenience.

$$\mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.5a}$$

As before, we substitute the complex number notation for the vectors, denoting their scalar lengths as *a, b, c, d* as shown in Figure 7-5.

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0 \tag{4.5c}$$

In Section 6.7 (p. 271), we differentiated equation 4.5c versus time to get an expression for velocity which is repeated here.

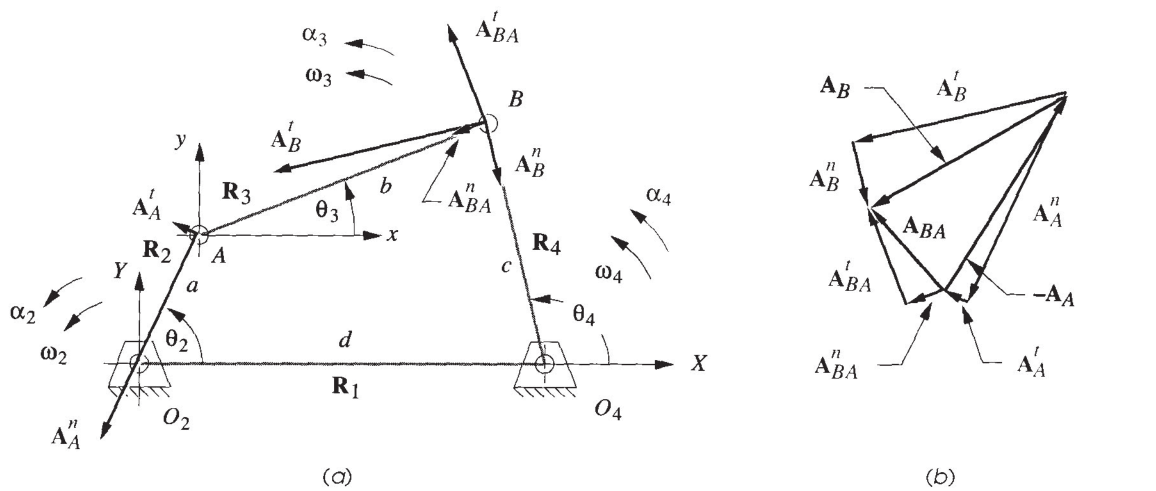


FIGURE 7-5

Position vector loop for a fourbar linkage showing acceleration vectors

$$ja\omega_2 e^{j\theta_2} + jb\omega_3 e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0 \tag{6.14c}$$

We will now differentiate equation 6.14c versus time to obtain an expression for accelerations in the linkage. Each term in equation 6.14c contains two functions of time, θ and ω. Differentiating with the chain rule in this example will result in two terms in the acceleration expression for each term in the velocity equation.

$$\left(j^2 a\omega_2^2 e^{j\theta_2} + ja\alpha_2 e^{j\theta_2}\right) + \left(j^2 b\omega_3^2 e^{j\theta_3} + jb\alpha_3 e^{j\theta_3}\right) - \left(j^2 c\omega_4^2 e^{j\theta_4} + jc\alpha_4 e^{j\theta_4}\right) = 0 \tag{7.7a}$$

Simplifying and grouping terms:

$$\left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) + \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) - \left(c\alpha_4 je^{j\theta_4} - c\omega_4^2 e^{j\theta_4}\right) = 0 \tag{7.7b}$$

Compare the terms grouped in parentheses with equations 7.2 (p. 301). Equation 7.7 contains the tangential and normal components of the accelerations of points A and B and of the acceleration difference of B to A. Note that these are the same relationships which we used to solve this problem graphically in Section 7.2 (p. 303). Equation 7.7 is, in fact, the **acceleration difference equation** 7.4 which, with the labels used here, is:

$$\mathbf{A}_A + \mathbf{A}_{BA} - \mathbf{A}_B = 0 \tag{7.8a}$$

where:

$$\begin{aligned}
\mathbf{A}_A &= \left(\mathbf{A}_A^t + \mathbf{A}_A^n\right) = \left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) \\
\mathbf{A}_{BA} &= \left(\mathbf{A}_{BA}^t + \mathbf{A}_{BA}^n\right) = \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) \\
\mathbf{A}_B &= \left(\mathbf{A}_B^t + \mathbf{A}_B^n\right) = \left(c\alpha_4 je^{j\theta_4} - c\omega_4^2 e^{j\theta_4}\right)
\end{aligned} \tag{7.8b}$$

The vector diagram in Figure 7-5b (p. 309) shows these components and is a graphical solution to equation 7.8a. The vector components are also shown acting at their respective points on Figure 7-5a.

We now need to solve equation 7.7 for α_3 and α_4, knowing the input angular acceleration α_2, the link lengths, all link angles, and angular velocities. Thus, the position analysis derived in Section 4.5 (p. 152) and the velocity analysis from Section 6.7 (p. 271) must be done first to determine the link angles and angular velocities before this acceleration analysis can be completed. We wish to solve equation 7.8 to get expressions in this form:

$$\alpha_3 = f(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2) \tag{7.9a}$$

$$\alpha_4 = g(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2) \tag{7.9b}$$

The strategy of solution will be the same as was done for the position and velocity analysis. First, substitute the Euler identity from equation 4.4a in each term of equation 7.7:

$$\begin{aligned}&\left[a\alpha_2\, j(\cos\theta_2 + j\sin\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2)\right]\\ &\quad+\left[b\alpha_3\, j(\cos\theta_3 + j\sin\theta_3) - b\omega_3^2(\cos\theta_3 + j\sin\theta_3)\right]\\ &\qquad-\left[c\alpha_4\, j(\cos\theta_4 + j\sin\theta_4) - c\omega_4^2(\cos\theta_4 + j\sin\theta_4)\right] = 0\end{aligned} \tag{7.10a}$$

Multiply by the operator j and rearrange:

$$\begin{aligned}&\left[a\alpha_2(-\sin\theta_2 + j\cos\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2)\right]\\ &\quad+\left[b\alpha_3(-\sin\theta_3 + j\cos\theta_3) - b\omega_3^2(\cos\theta_3 + j\sin\theta_3)\right]\\ &\qquad-\left[c\alpha_4(-\sin\theta_4 + j\cos\theta_4) - c\omega_4^2(\cos\theta_4 + j\sin\theta_4)\right] = 0\end{aligned} \tag{7.10b}$$

We can now separate this vector equation into its two components by collecting all real and all imaginary terms separately:

real part (x component):

$$-a\alpha_2\sin\theta_2 - a\omega_2^2\cos\theta_2 - b\alpha_3\sin\theta_3 - b\omega_3^2\cos\theta_3 + c\alpha_4\sin\theta_4 + c\omega_4^2\cos\theta_4 = 0 \tag{7.11a}$$

imaginary part (y component):

$$a\alpha_2\cos\theta_2 - a\omega_2^2\sin\theta_2 + b\alpha_3\cos\theta_3 - b\omega_3^2\sin\theta_3 - c\alpha_4\cos\theta_4 + c\omega_4^2\sin\theta_4 = 0 \tag{7.11b}$$

Note that the j's have cancelled in equation 7.11b. We can solve equations 7.11a and 7.11b simultaneously to get:

$$\alpha_3 = \frac{CD - AF}{AE - BD} \tag{7.12a}$$

$$\alpha_4 = \frac{CE - BF}{AE - BD} \tag{7.12b}$$

where:

$$
\begin{aligned}
A &= c\sin\theta_4 \\
B &= b\sin\theta_3 \\
C &= a\alpha_2\sin\theta_2 + a\omega_2^2\cos\theta_2 + b\omega_3^2\cos\theta_3 - c\omega_4^2\cos\theta_4 \\
D &= c\cos\theta_4 \\
E &= b\cos\theta_3 \\
F &= a\alpha_2\cos\theta_2 - a\omega_2^2\sin\theta_2 - b\omega_3^2\sin\theta_3 + c\omega_4^2\sin\theta_4
\end{aligned}
\tag{7.12c}
$$

Once we have solved for α_3 and α_4, we can then solve for the linear accelerations by substituting the Euler identity into equations 7.8b,

$$\mathbf{A}_A = a\alpha_2\left(-\sin\theta_2 + j\cos\theta_2\right) - a\omega_2^2\left(\cos\theta_2 + j\sin\theta_2\right) \tag{7.13a}$$

$$\mathbf{A}_{BA} = b\alpha_3\left(-\sin\theta_3 + j\cos\theta_3\right) - b\omega_3^2\left(\cos\theta_3 + j\sin\theta_3\right) \tag{7.13b}$$

$$\mathbf{A}_B = c\alpha_4\left(-\sin\theta_4 + j\cos\theta_4\right) - c\omega_4^2\left(\cos\theta_4 + j\sin\theta_4\right) \tag{7.13c}$$

where the real and imaginary terms are the x and y components, respectively. Equations 7.12 and 7.13 provide a complete solution for the angular accelerations of the links and the linear accelerations of the joints in the pin-jointed fourbar linkage.

The Fourbar Slider-Crank

The first inversion of the offset slider-crank has its slider block sliding against the ground plane as shown in Figure 7-6a. Its accelerations can be solved for in similar manner as was done for the pin-jointed fourbar.

The position equations for the fourbar offset slider-crank linkage (inversion #1) were derived in Section 4.6 (p. 159). The linkage was shown in Figures 4-9 (p. 160) and 6-21 (p. 275) and is shown again in Figure 7-6a on which we also show an input angular acceleration α_2 applied to link 2. This α_2 can be a time-varying input acceleration. The vector loop equation 4.14 is repeated here for your convenience.

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.14a}$$

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \tag{4.14b}$$

In Section 6.7 (p. 267) we differentiated equation 4.14b with respect to time noting that a, b, c, θ_1, and θ_4 are constant but the length of link d varies with time in this inversion.

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{d} = 0 \tag{6.20a}$$

The term *d dot* is the linear velocity of the slider block. Equation 6.20a is the velocity difference equation.

We now will differentiate equation 6.20a with respect to time to get an expression for acceleration in this inversion of the slider-crank mechanism.

$$\left(ja\alpha_2 e^{j\theta_2} + j^2 a\omega_2^2 e^{j\theta_2}\right) - \left(jb\alpha_3 e^{j\theta_3} + j^2 b\omega_3^2 e^{j\theta_3}\right) - \ddot{d} = 0 \tag{7.14a}$$

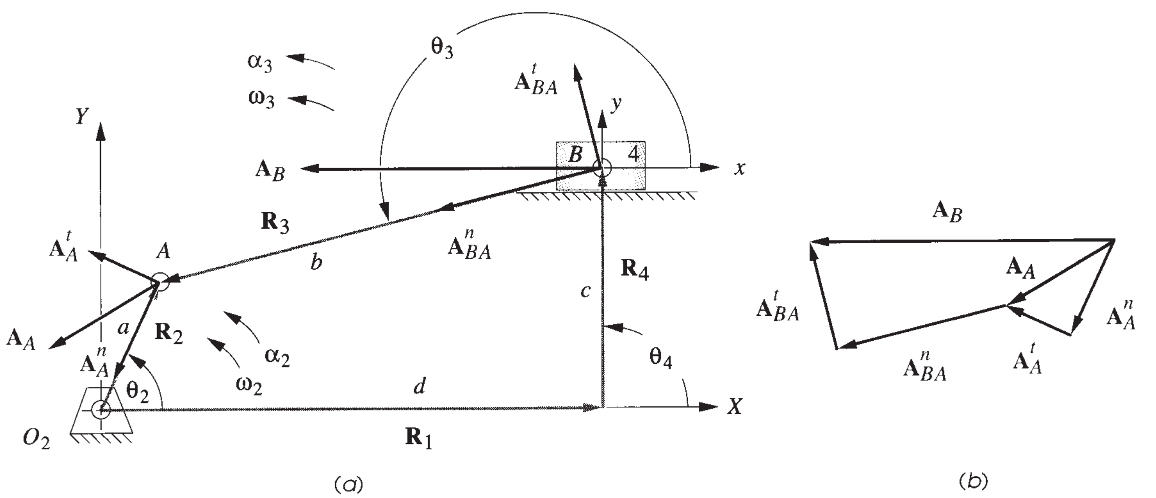


FIGURE 7-6

Position vector loop for a fourbar slider-crank linkage showing acceleration vectors

Simplifying:

$$\left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) - \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) - \ddot{d} = 0 \tag{7.14b}$$

Note that equation 7.14 is again the acceleration difference equation:

$$\begin{aligned} \mathbf{A}_A - \mathbf{A}_{AB} - \mathbf{A}_B &= 0 \\ \mathbf{A}_{BA} &= -\mathbf{A}_{AB} \\ \mathbf{A}_B &= \mathbf{A}_A + \mathbf{A}_{BA} \end{aligned} \tag{7.15a}$$

$$\begin{aligned} \mathbf{A}_A &= \left(\mathbf{A}_A^t + \mathbf{A}_A^n\right) = \left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) \\ \mathbf{A}_{BA} &= \left(\mathbf{A}_{BA}^t + \mathbf{A}_{BA}^n\right) = \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right) \\ \mathbf{A}_B &= \mathbf{A}_B^t = \ddot{d} \end{aligned} \tag{7.15b}$$

Note that in this mechanism, link 4 is in pure translation and so has zero ω_4 and zero α_4. The acceleration of link 4 has only a "tangential" component of acceleration along its path.

The two unknowns in the vector equation 7.14 are the angular acceleration of link 3, α_3, and the linear acceleration of link 4, *d double dot*. To solve for them, substitute the Euler identity,

$$\begin{aligned} &a\alpha_2\left(-\sin\theta_2 + j\cos\theta_2\right) - a\omega_2^2\left(\cos\theta_2 + j\sin\theta_2\right) \\ &\quad - b\alpha_3\left(-\sin\theta_3 + j\cos\theta_3\right) + b\omega_3^2\left(\cos\theta_3 + j\sin\theta_3\right) - \ddot{d} = 0 \end{aligned} \tag{7.16a}$$

and separate the real (x) and imaginary (y) components:

real part (x component):

$$-a\alpha_2 \sin\theta_2 - a\omega_2^2 \cos\theta_2 + b\alpha_3 \sin\theta_3 + b\omega_3^2 \cos\theta_3 - \ddot{d} = 0 \tag{7.16b}$$

imaginary part (y component):

$$a\alpha_2 \cos\theta_2 - a\omega_2^2 \sin\theta_2 - b\alpha_3 \cos\theta_3 + b\omega_3^2 \sin\theta_3 = 0 \tag{7.16c}$$

Equation 7.16c can be solved directly for α_3 and the result substituted in equation 7.16b to find *d double dot.*

$$\alpha_3 = \frac{a\alpha_2 \cos\theta_2 - a\omega_2^2 \sin\theta_2 + b\omega_3^2 \sin\theta_3}{b\cos\theta_3} \tag{7.16d}$$

$$\ddot{d} = -a\alpha_2 \sin\theta_2 - a\omega_2^2 \cos\theta_2 + b\alpha_3 \sin\theta_3 + b\omega_3^2 \cos\theta_3 \tag{7.16e}$$

The other linear accelerations can be found from equation 7.15b and are shown in the vector diagram of Figure 7-6b.

Coriolis Acceleration

The examples used for acceleration analysis above have involved only pin-jointed linkages or the inversion of the slider-crank in which the slider block has no rotation. When a sliding joint is present on a rotating link, an additional component of acceleration will be present, called the **Coriolis component**, after its discoverer. Figure 7-7a shows a simple, two-link system consisting of a link with a radial slot, and a slider block free to slip within that slot.

The instantaneous location of the block is defined by a position vector ($\mathbf{R}_P$) referenced to the global origin at the link center. *This vector is both rotating and changing length as the system moves.* As shown this is a two-degree-of-freedom system. The **two inputs to the system** are the angular acceleration (α) of the link and the relative linear slip velocity ($\mathbf{V}_{P_{slip}}$) of the block versus the disk. The angular velocity ω is a result of the time history of the angular acceleration. The situation shown, with a counterclockwise α and a clockwise ω, implies that earlier in time the link had been accelerated up to a clockwise angular velocity and is now being slowed down. The transmission component of velocity ($\mathbf{V}_{P_{trans}}$) is a result of the ω of the link acting at the radius R_P whose magnitude is p.

We show the situation in Figure 7-7 at one instant of time. However, the equations to be derived will be valid for all time. We want to determine the acceleration at the center of the block (P) under this combined motion of rotation and sliding. To do so we first write the expression for the position vector $\mathbf{R}_P$ which locates point P.

$$\mathbf{R}_P = pe^{j\theta_2} \tag{7.17}$$

Note that there are two functions of time in equation 7.17, p and θ. When we differentiate versus time we get two terms in the velocity expression:

$$\mathbf{V}_P = p\omega_2 je^{j\theta_2} + \dot{p}e^{j\theta_2} \tag{7.18a}$$

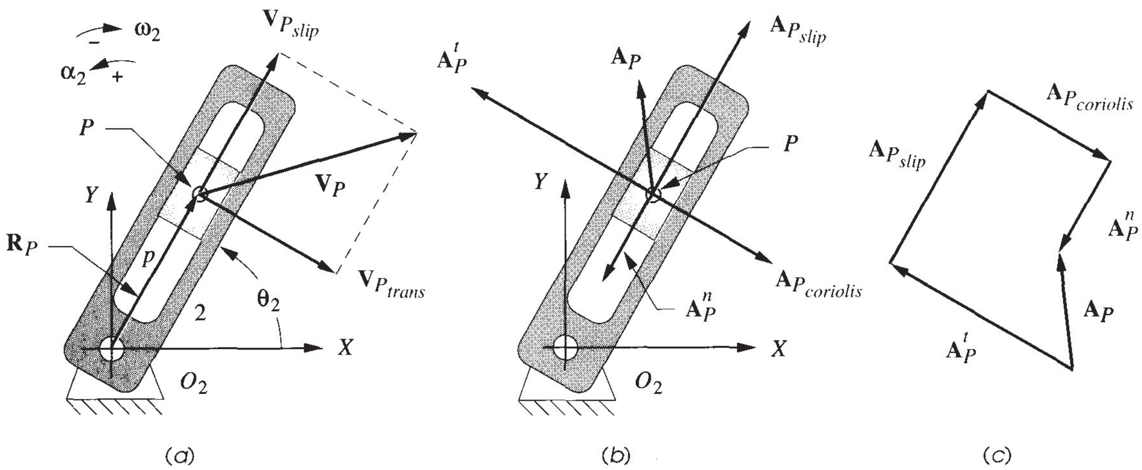


FIGURE 7-7

The Coriolis component of acceleration shown in a system with a positive (CCW) α_2 and a negative (CW) ω_2

These are the transmission component and the slip component of velocity.

$$\mathbf{V}_P = \mathbf{V}_{P_{trans}} + \mathbf{V}_{P_{slip}} \tag{7.18b}$$

The $p\omega$ term is the transmission component and is directed at 90 degrees to the axis of slip which, in this example, is coincident with the position vector $\mathbf{R}_P$. The *p dot* term is the **slip component** and is directed along the **axis of slip** in the same direction as the position vector in this example. Their vector sum is $\mathbf{V}_P$ as shown in Figure 7-7a.

To get an expression for acceleration, we must differentiate equation 7.18 versus time. Note that the transmission component has **three** functions of time in it, p, ω, and θ. The chain rule will yield three terms for this one term. The slip component of velocity contains two functions of time, p and θ, yielding two terms in the derivative for a total of five terms, two of which turn out to be the same.

$$\mathbf{A}_P = \left(p\alpha_2 je^{j\theta_2} + p\omega_2^2 j^2 e^{j\theta_2} + \dot{p}\omega_2 je^{j\theta_2}\right) + \left(\dot{p}\omega_2 je^{j\theta_2} + \ddot{p}e^{j\theta_2}\right) \tag{7.19a}$$

Simplifying and collecting terms:

$$\mathbf{A}_P = p\alpha_2 je^{j\theta_2} - p\omega_2^2 e^{j\theta_2} + 2\dot{p}\omega_2 je^{j\theta_2} + \ddot{p}e^{j\theta_2} \tag{7.19b}$$

These terms represent the following components:

$$\mathbf{A}_P = \mathbf{A}_{P_{tangential}} + \mathbf{A}_{P_{normal}} + \mathbf{A}_{P_{coriolis}} + \mathbf{A}_{P_{slip}} \tag{7.19c}$$

Note that the Coriolis term has appeared in the acceleration expression as a result of the differentiation simply because the length of the vector p is a function of time. The Coriolis component magnitude is twice the product of the velocity of slip (equation 7.18) and the angular velocity of the link containing the slider slot. Its direction is rotated 90 degrees from that of the original position vector $\mathbf{R}_P$ either clockwise or counterclockwise depending on the sense of 0). (Note that we chose to align the position vector R_P with the axis of slip in Figure 7-7 which can always be done regardless of the location of the center of rotation-also see Figure 7-6 (p. 312) where **RJ** is aligned with the axis of slip.) All four components from equation 7.19 are shown acting on point P in Figure 7-7b. The total acceleration A_P is the vector sum of the four terms as shown in Figure 7-7c. Note that the normal acceleration term in equation 7.19b is negative in sign, so it becomes a subtraction when substituted in equation 7.19c.

This Coriolis component of acceleration will always be present when there is a velocity of slip associated with any member which also has an angular velocity. In the absence of either of those two factors the Coriolis component will be zero. You have probably experienced Coriolis acceleration if you have ever ridden on a carousel or merry-go-round. If you attempted to walk radially from the outside to the inside (or vice versa) while the carousel was turning, you were thrown sideways by the inertial force due to the Coriolis acceleration. You were the *slider block* in Figure 7-7, and your *slip velocity* combined with the rotation of the carousel created the Coriolis component. As you walked from a large radius to a smaller one, your tangential velocity had to change to match that of the new location of your foot on the spinning carousel. Any change in velocity requires an acceleration to accomplish. It was the *"ghost of Coriolis"* that pushed you sideways on that carousel.

Another example of the Coriolis component is its effect on weather systems. Large objects which exist in the earth's lower atmosphere, such as hurricanes, span enough area to be subject to significantly different velocities at their northern and southern extremities. The atmosphere turns with the earth. The earth's surface tangential velocity due to its angular velocity varies from zero at the poles to a maximum of about 1000 mph at the equator. The winds of a storm system are attracted toward the low pressure at its center. These winds have a slip velocity with respect to the surface, which in combination with the earth's 0), creates a Coriolis component of acceleration on the moving air masses. This Coriolis acceleration causes the inrushing air to rotate about the center, or "eye" of the storm system. This rotation will be counterclockwise in the northei-n hemisphere and clockwise in the southern hemisphere. The movement of the entire storm system from south to north also creates a Coriolis component which will tend to deviate the storm's track eastward, though this effect is often overridden by the forces due to other large air masses such as high-pressure systems which can deflect a storm. These complicated factors make it difficult to predict a large storm's true track.

Note that in the analytical solution presented here, the Coriolis component will be accounted for automatically as long as the differentiations are correctly done. However, when doing a graphical acceleration analysis one must be on the alert to recognize the presence of this component, calculate it, and include it in the vector diagrams when its two constituents V_{slip} and 0) are both nonzero.

The Fourbar Inverted Slider-Crank

The position equations for the fourbar inverted slider-crank linkage were derived in Section 4.7 (p. 159). The linkage was shown in Figures 4-10 (p. 162) and 6-22 (p. 277) and is shown again in Figure 7-8a on which we also show an input angular acceleration a2 applied to link 2. This a2 can vary with time. The vector loop equations 4.14 (p. 311) are valid for this linkage as well.

All slider linkages will have at least one link whose effective length between joints varies as the linkage moves. In this inversion the length of link 3 between points A and B, designated as b, will change as it passes through the slider block on link 4. In Section 6.7 (p. 267) we got an expression for velocity, by differentiating equation 4.14b with respect to time noting that a, c, d, and θ_1 are constant and b varies with time.

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{b}e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0 \tag{6.24}$$

Differentiating this with respect to time will give an expression for accelerations in this inversion of the slider-crank mechanism.

$$\begin{aligned}\left(ja\alpha_2 e^{j\theta_2} + j^2 a\omega_2^2 e^{j\theta_2}\right) - \left(jb\alpha_3 e^{j\theta_3} + j^2 b\omega_3^2 e^{j\theta_3} + j\dot{b}\omega_3 e^{j\theta_3}\right)& \\ - \left(\ddot{b}e^{j\theta_3} + j\dot{b}\omega_3 e^{j\theta_3}\right) - \left(jc\alpha_4 e^{j\theta_4} + j^2 c\omega_4^2 e^{j\theta_4}\right) = 0&\end{aligned} \tag{7.20a}$$

Simplifying and collecting terms:

$$\begin{aligned}\left(a\alpha_2\, je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) - \left(b\alpha_3\, je^{j\theta_3} - b\omega_3^2 e^{j\theta_3} + 2\dot{b}\omega_3\, je^{j\theta_3} + \ddot{b}e^{j\theta_3}\right)& \\ - \left(c\alpha_4\, je^{j\theta_4} - c\omega_4^2 e^{j\theta_4}\right) = 0&\end{aligned} \tag{7.20b}$$

Equation 7.20 is in fact the acceleration difference equation (equation 7.4, p. 302) and can be written in that notation as shown in equation 7.21.

$$\begin{aligned}&\mathbf{A}_A - \mathbf{A}_{AB} - \mathbf{A}_B = 0\\ \text{but:}\quad &\mathbf{A}_{BA} = -\mathbf{A}_{AB}\\ \text{and:}\quad &\mathbf{A}_B = \mathbf{A}_A + \mathbf{A}_{BA}\end{aligned} \tag{7.21a}$$

$$\begin{aligned}\mathbf{A}_A &= \mathbf{A}_{A_{tangential}} + \mathbf{A}_{A_{normal}}\\ \mathbf{A}_{AB} &= \mathbf{A}_{AB_{tangential}} + \mathbf{A}_{AB_{normal}} + \mathbf{A}_{AB_{coriolis}} + \mathbf{A}_{AB_{slip}}\\ \mathbf{A}_B &= \mathbf{A}_{B_{tangential}} + \mathbf{A}_{B_{normal}}\end{aligned} \tag{7.21b}$$

$$\begin{aligned}\mathbf{A}_{A_{tangential}} &= a\alpha_2\, je^{j\theta_2} & \mathbf{A}_{A_{normal}} &= -a\omega_2^2 e^{j\theta_2}\\ \mathbf{A}_{B_{tangential}} &= c\alpha_4\, je^{j\theta_4} & \mathbf{A}_{B_{normal}} &= -c\omega_4^2 e^{j\theta_4}\\ \mathbf{A}_{AB_{tangential}} &= b\alpha_3\, je^{j\theta_3} & \mathbf{A}_{AB_{normal}} &= -b\omega_3^2 e^{j\theta_3}\\ \mathbf{A}_{AB_{coriolis}} &= 2\dot{b}\omega_3\, je^{j\theta_3} & \mathbf{A}_{AB_{slip}} &= \ddot{b}e^{j\theta_3}\end{aligned} \tag{7.21c}$$

Because this sliding link also has an angular velocity, there will be a nonzero Coriolis component of acceleration at point B which is the 2 *b dot* term in equation 7.20. Since a complete velocity analysis was done before doing this acceleration analysis, the Coriolis component can be readily calculated at this point, knowing both ω and $\mathbf{V}_{slip}$ from the velocity analysis.

The *b double dot* term in equation 7.21a is the *slip component of acceleration.* This is one of the variables to be solved for in this acceleration analysis. Another variable to

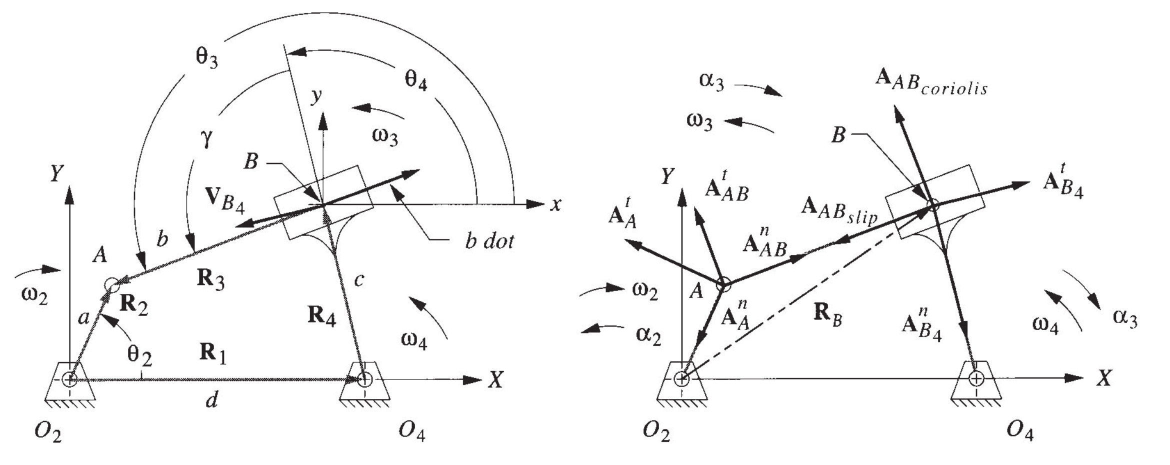


FIGURE 7-8

Acceleration analysis of inversion #3 of the fourbar slider-crank driven with a positive (CCW) α_2 and a negative (CW) ω_2

be solved for is α_4, the angular acceleration of link 4. Note, however, that we also have an unknown in α_3, the angular acceleration of link 3. This is a total of three unknowns. Equation 7.20 can only be solved for two unknowns. Thus we require another equation to solve the system. There is a fixed relationship between angles θ_3 and θ_4, shown as γ in Figure 7-8 and defined in equation 4.18, repeated here:

$$\theta_3 = \theta_4 \pm \gamma \tag{4.18}$$

Differentiate it twice with respect to time to obtain:

$$\omega_3 = \omega_4; \qquad\qquad \alpha_3 = \alpha_4 \tag{7.22}$$

We wish to solve equation 7.20 to get expressions in this form:

$$\alpha_3 = \alpha_4 = f\left(a, b, \dot{b}, c, d, \theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2\right) \tag{7.23a}$$

$$\frac{d^2 b}{dt^2} = \ddot{b} = g\left(a, b, \dot{b}, c, d, \theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2\right) \tag{7.23b}$$

Substitution of the Euler identity (equation 4.4a, p. 155) into equation 7.20 yields:

$$\begin{aligned} &a\alpha_2\, j(\cos\theta_2 + j\sin\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2) \\ &\quad - b\alpha_3\, j(\cos\theta_3 + j\sin\theta_3) + b\omega_3^2(\cos\theta_3 + j\sin\theta_3) \\ &\quad\quad - 2\dot{b}\omega_3\, j(\cos\theta_3 + j\sin\theta_3) - \ddot{b}(\cos\theta_3 + j\sin\theta_3) \\ &\quad\quad\quad - c\alpha_4\, j(\cos\theta_4 + j\sin\theta_4) + c\omega_4^2(\cos\theta_4 + j\sin\theta_4) = 0 \end{aligned} \tag{7.24a}$$

Multiply by the operator j and substitute α_4 for α_3 from equation 7.22:

$$\begin{aligned} a\alpha_2(-\sin\theta_2 + j\cos\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2) \\ - b\alpha_4(-\sin\theta_3 + j\cos\theta_3) + b\omega_3^2(\cos\theta_3 + j\sin\theta_3) \\ - 2\dot{b}\omega_3(-\sin\theta_3 + j\cos\theta_3) - \ddot{b}(\cos\theta_3 + j\sin\theta_3) \\ - c\alpha_4(-\sin\theta_4 + j\cos\theta_4) + c\omega_4^2(\cos\theta_4 + j\sin\theta_4) = 0 \end{aligned} \tag{7.24b}$$

We can now separate this vector equation 7.24b into its two components by collecting all real and all imaginary terms separately:

real part (x component):

$$\begin{aligned} -a\alpha_2\sin\theta_2 - a\omega_2^2\cos\theta_2 + b\alpha_4\sin\theta_3 + b\omega_3^2\cos\theta_3 \\ + 2\dot{b}\omega_3\sin\theta_3 - \ddot{b}\cos\theta_3 + c\alpha_4\sin\theta_4 + c\omega_4^2\cos\theta_4 = 0 \end{aligned} \tag{7.25a}$$

imaginary part (y component):

$$\begin{aligned} a\alpha_2\cos\theta_2 - a\omega_2^2\sin\theta_2 - b\alpha_4\cos\theta_3 + b\omega_3^2\sin\theta_3 \\ - 2\dot{b}\omega_3\cos\theta_3 - \ddot{b}\sin\theta_3 - c\alpha_4\cos\theta_4 + c\omega_4^2\sin\theta_4 = 0 \end{aligned} \tag{7.25b}$$

Note that the j's have cancelled in equation 7.25b. We can solve equations 7.25 simultaneously for the two unknowns, α_4 and b *double dot.* The solution is:

$$\alpha_4 = \frac{a\left[\alpha_2\cos(\theta_3 - \theta_2) + \omega_2^2\sin(\theta_3 - \theta_2)\right] + c\omega_4^2\sin(\theta_4 - \theta_3) - 2\dot{b}\omega_3}{b + c\cos(\theta_3 - \theta_4)} \tag{7.26a}$$

$$\ddot{b} = -\frac{\left\{\begin{aligned} a\omega_2^2\left[b\cos(\theta_3 - \theta_2) + c\cos(\theta_4 - \theta_2)\right] + a\alpha_2\left[b\sin(\theta_2 - \theta_3) - c\sin(\theta_4 + \theta_2)\right] \\ + 2\dot{b}c\omega_4\sin(\theta_4 - \theta_3) - \omega_4^2\left[b^2 + c^2 + 2bc\cos(\theta_4 - \theta_3)\right] \end{aligned}\right\}}{b + c\cos(\theta_3 - \theta_4)} \tag{7.26b}$$

Equation 7.26a provides the **angular acceleration** of link 4. Equation 7.26b provides the **acceleration of slip** at point B. Once these variables are solved for, the linear accelerations at points A and B in the linkage of Figure 7-8 (p. 317) can be found by substituting the Euler identity into equations 7.21.

$$\mathbf{A}_A = a\alpha_2(-\sin\theta_2 + j\cos\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2) \tag{7.27a}$$

$$\begin{aligned} \mathbf{A}_{BA} = b\alpha_3(\sin\theta_3 - j\cos\theta_3) + b\omega_3^2(\cos\theta_3 + j\sin\theta_3) \\ + 2\dot{b}\omega_3(\sin\theta_3 - j\cos\theta_3) - \ddot{b}(\cos\theta_3 + j\sin\theta_3) \end{aligned} \tag{7.27b}$$

$$\mathbf{A}_B = -c\alpha_4(\sin\theta_4 - j\cos\theta_4) - c\omega_4^2(\cos\theta_4 + j\sin\theta_4) \tag{7.27c}$$

These components of these vectors are shown in Figure 7-8b.

7.4 ACCELERATION ANALYSIS OF THE GEARED FIVEBAR LINKAGE

The velocity equation for the geared fivebar mechanism was derived in Section 6.8 (p. 317) and is repeated here. See Figure P7-4 (p. 331) for notation.

$$a\omega_2 je^{j\theta_2} + b\omega_3 je^{j\theta_3} - c\omega_4 je^{j\theta_4} - d\omega_5 je^{j\theta_5} = 0 \tag{6.32a}$$

Differentiate this with respect to time to get an expression for acceleration.

$$\begin{aligned}&\left(a\alpha_2 je^{j\theta_2} - a\omega_2^2 e^{j\theta_2}\right) + \left(b\alpha_3 je^{j\theta_3} - b\omega_3^2 e^{j\theta_3}\right)\\ &\quad -\left(c\alpha_4 je^{j\theta_4} - c\omega_4^2 e^{j\theta_4}\right) - \left(d\alpha_5 je^{j\theta_5} - d\omega_5^2 e^{j\theta_5}\right) = 0\end{aligned} \tag{7.28a}$$

Substitute the Euler equivalents:

$$\begin{aligned}&a\alpha_2(-\sin\theta_2 + j\cos\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2)\\ &\quad + b\alpha_3(-\sin\theta_3 + j\cos\theta_3) - b\omega_3^2(\cos\theta_3 + j\sin\theta_3)\\ &\quad - c\alpha_4(-\sin\theta_4 + j\cos\theta_4) + c\omega_4^2(\cos\theta_4 + j\sin\theta_4)\\ &\quad - d\alpha_5(-\sin\theta_5 + j\cos\theta_5) + d\omega_5^2(\cos\theta_5 + j\sin\theta_5) = 0\end{aligned} \tag{7.28b}$$

Note that the angle θ_5 is defined in terms of θ_2, the gear ratio λ, and the phase angle ϕ. This relationship and its derivatives are:

$$\theta_5 = \lambda\theta_2 + \phi; \qquad \omega_5 = \lambda\omega_2; \qquad \alpha_5 = \lambda\alpha_2 \tag{7.28c}$$

Since a complete position and velocity analysis must be done before an acceleration analysis, we will assume that the values of θ_5 and ω_5 have been found and will leave these equations in terms of θ_5, ω_5, and α_5.

Separating the real and imaginary terms in equation 7.28b:

real:

$$\begin{aligned}&-a\alpha_2\sin\theta_2 - a\omega_2^2\cos\theta_2 - b\alpha_3\sin\theta_3 - b\omega_3^2\cos\theta_3\\ &\quad + c\alpha_4\sin\theta_4 + c\omega_4^2\cos\theta_4 + d\alpha_5\sin\theta_5 + d\omega_5^2\cos\theta_5 = 0\end{aligned} \tag{7.28d}$$

imaginary:

$$\begin{aligned}&a\alpha_2\cos\theta_2 - a\omega_2^2\sin\theta_2 + b\alpha_3\cos\theta_3 - b\omega_3^2\sin\theta_3\\ &\quad - c\alpha_4\cos\theta_4 + c\omega_4^2\sin\theta_4 - d\alpha_5\cos\theta_5 + d\omega_5^2\sin\theta_5 = 0\end{aligned} \tag{7.28e}$$

The only two unknowns are α_3 and α_4. Either equation 7.28d or 7.28e can be solved for one unknown and the result substituted in the other. The solution for α_3 is:

$$\alpha_3 = \frac{\begin{bmatrix} -a\alpha_2 \sin(\theta_2 - \theta_4) - a\omega_2^2 \cos(\theta_2 - \theta_4) \\ -b\omega_3^2 \cos(\theta_3 - \theta_4) + d\omega_5^2 \cos(\theta_5 - \theta_4) \\ + d\alpha_5 \sin(\theta_5 - \theta_4) + c\omega_4^2 \end{bmatrix}}{b\sin(\theta_3 - \theta_4)} \tag{7.29a}$$

and angle α_4 is:

$$\alpha_4 = \frac{\begin{bmatrix} a\alpha_2 \sin(\theta_2 - \theta_3) + a\omega_2^2 \cos(\theta_2 - \theta_3) \\ -c\omega_4^2 \cos(\theta_3 - \theta_4) - d\omega_5^2 \cos(\theta_3 - \theta_5) \\ + d\alpha_5 \sin(\theta_3 - \theta_5) + b\omega_3^2 \end{bmatrix}}{c\sin(\theta_4 - \theta_3)} \tag{7.29b}$$

With all link angles, angular velocities, and angular accelerations known, the linear accelerations for the pin joints can be found from:

$$\mathbf{A}_A = a\alpha_2(-\sin\theta_2 + j\cos\theta_2) - a\omega_2^2(\cos\theta_2 + j\sin\theta_2) \tag{7.29c}$$

$$\mathbf{A}_{BA} = b\alpha_3(-\sin\theta_3 + j\cos\theta_3) - b\omega_3^2(\cos\theta_3 + j\sin\theta_3) \tag{7.29d}$$

$$\mathbf{A}_C = c\alpha_5(-\sin\theta_5 + j\cos\theta_5) - c\omega_5^2(\cos\theta_5 + j\sin\theta_5) \tag{7.29e}$$

$$\mathbf{A}_B = \mathbf{A}_A + \mathbf{A}_{BA} \tag{7.29f}$$

7.5 ACCELERATION OF ANY POINT ON A LINKAGE

Once the angular accelerations of all the links are found it is easy to define and calculate the acceleration of *any point on any link* for any input position of the linkage. Figure 7-9 shows the fourbar linkage with its coupler, link 3, enlarged to contain a coupler point P. The crank and rocker have also been enlarged to show points S and U which might represent the centers of gravity of those links. We want to develop algebraic expressions for the accelerations of these (or any) points on the links.

To find the acceleration of point S, draw the position vector from the fixed pivot O_2 to point S. This vector $\mathbf{R}_{SO_2}$ makes an angle δ_2 with the vector $\mathbf{R}_{AO_2}$. This angle δ_2 is completely defined by the geometry of link 2 and is constant. The position vector for point S is then:

$$\mathbf{R}_{SO_2} = \mathbf{R}_S = se^{j(\theta_2 + \delta_2)} = s\left[\cos(\theta_2 + \delta_2) + j\sin(\theta_2 + \delta_2)\right] \tag{4.25}$$

We differentiated this position vector in Section 6.9 (p. 279) to find the velocity of that point. The equation is repeated here for your convenience.

$$\mathbf{V}_S = jse^{j(\theta_2 + \delta_2)}\omega_2 = s\omega_2\left[-\sin(\theta_2 + \delta_2) + j\cos(\theta_2 + \delta_2)\right] \tag{6.34}$$

We can differentiate again versus time to find the acceleration of point S.

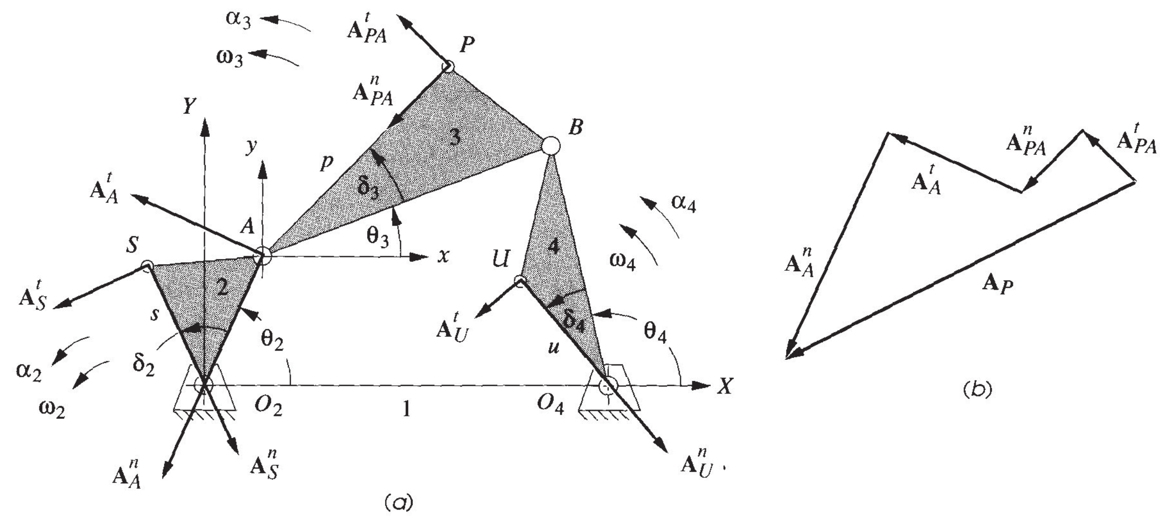


FIGURE 7-9
Finding the acceleration of any point on any link

$$
\begin{aligned}
\mathbf{A}_S &= s\alpha_2 \, j e^{j(\theta_2+\delta_2)} - s\omega_2^2 \, e^{j(\theta_2+\delta_2)} \\
&= s\alpha_2\left[-\sin(\theta_2+\delta_2)+j\cos(\theta_2+\delta_2)\right] \\
&\quad - s\omega_2^2\left[\cos(\theta_2+\delta_2)+j\sin(\theta_2+\delta_2)\right]
\end{aligned} \tag{7.30}
$$

The position of point U on link 4 is found in the same way, using the angle δ_4 which is a constant angular offset within the link. The expression is:

$$
\mathbf{R}_{UO_4} = u e^{j(\theta_4+\delta_4)} = u\left[\cos(\theta_4+\delta_4)+j\sin(\theta_4+\delta_4)\right] \tag{4.26}
$$

We differentiated this position vector in Section 6.9 to find the velocity of that point. The equation is repeated here for your convenience.

$$
\mathbf{V}_U = j u e^{j(\theta_4+\delta_4)}\omega_4 = u\omega_4\left[-\sin(\theta_4+\delta_4)+j\cos(\theta_4+\delta_4)\right] \tag{6.35}
$$

We can differentiate again versus time to find the acceleration of point U.

$$
\begin{aligned}
\mathbf{A}_U &= u\alpha_4 \, j e^{j(\theta_4+\delta_4)} - u\omega_4^2 \, e^{j(\theta_4+\delta_4)} \\
&= u\alpha_4\left[-\sin(\theta_4+\delta_4)+j\cos(\theta_4+\delta_4)\right] \\
&\quad - u\omega_4^2\left[\cos(\theta_4+\delta_4)+j\sin(\theta_4+\delta_4)\right]
\end{aligned} \tag{7.31}
$$

The acceleration of point P on link 3 can be found from the addition of two acceleration vectors, such as $\mathbf{A}_A$ and $\mathbf{A}_{PA}$. Vector $\mathbf{A}_A$ is already defined from our analysis of the link accelerations. $\mathbf{A}_{PA}$ is the acceleration difference of point P with respect to point A. Point A is chosen as the reference point because angle θ_3 is defined at a local coordinate system whose origin is at A. Position vector $\mathbf{R}_{PA}$ is defined in the same way as $\mathbf{R}_U$ or $\mathbf{R}_S$, using the internal link offset angle δ_3 and the angle of link 3, θ_3. We previously analyzed this position vector and differentiated it in Section 6.9 to find the velocity difference of that point with respect to point A. Those equations are repeated here for your convenience.

$$\mathbf{R}_{PA} = pe^{j(\theta_3+\delta_3)} = p\left[\cos(\theta_3+\delta_3) + j\sin(\theta_3+\delta_3)\right] \tag{4.27a}$$

$$\mathbf{R}_P = \mathbf{R}_A + \mathbf{R}_{PA} \tag{4.27b}$$

$$\mathbf{V}_{PA} = jpe^{j(\theta_3+\delta_3)}\omega_3 = p\omega_3\left[-\sin(\theta_3+\delta_3) + j\cos(\theta_3+\delta_3)\right] \tag{6.36a}$$

$$\mathbf{V}_P = \mathbf{V}_A + \mathbf{V}_{PA} \tag{6.36b}$$

We can differentiate equation 6.36 again versus time to find $\mathbf{A}_{PA}$, the acceleration of point P versus A. This vector can then be added to the vector $\mathbf{A}_A$ already found to define the absolute acceleration $\mathbf{A}_P$ of point P.

$$\mathbf{A}_P = \mathbf{A}_A + \mathbf{A}_{PA} \tag{7.32a}$$

where:

$$\begin{aligned}\mathbf{A}_{PA} &= p\alpha_3\, je^{j(\theta_3+\delta_3)} - p\omega_3^2\, e^{j(\theta_3+\delta_3)} \\ &= p\alpha_3\left[-\sin(\theta_3+\delta_3) + j\cos(\theta_3+\delta_3)\right] \\ &\quad - p\omega_3^2\left[\cos(\theta_3+\delta_3) + j\sin(\theta_3+\delta_3)\right]\end{aligned} \tag{7.32b}$$

Please compare equation 7.32 with equation 7.4 (p. 302). It is again the acceleration difference equation. Note that this equation applies to **any point on any link** at any position for which the positions and velocities are defined. It is a general solution for any rigid body.

7.6 HUMAN TOLERANCE OF ACCELERATION

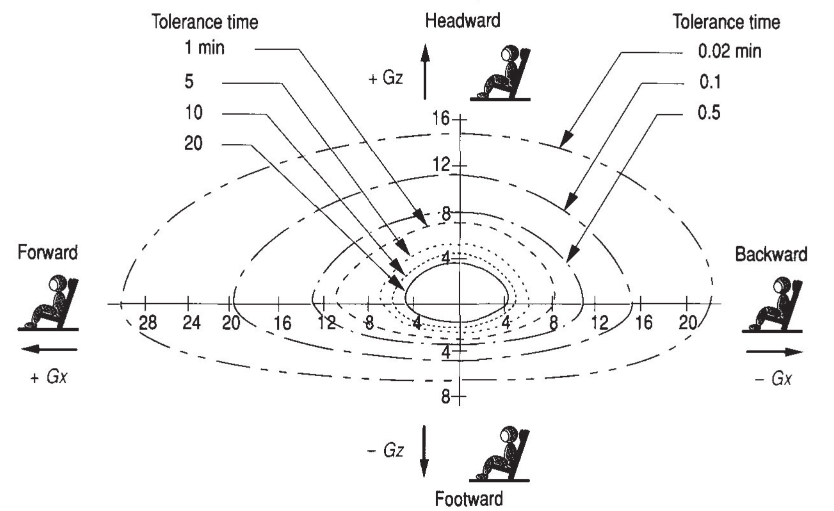
It is interesting to note that the human body does not sense velocity, except with the eyes, but is very sensitive to acceleration. Riding in an automobile, in the daylight, one can see the scenery passing by and have a sense of motion. But, traveling at night in a commercial airliner at a 500 mph constant velocity, we have no sensation of motion as long as the flight is smooth. What we will sense in this situation is any change in velocity due to atmospheric turbulence, takeoffs, or landings. The semicircular canals in the inner ear are sensitive accelerometers which report to us on any accelerations which we experience. You have no doubt also experienced the sensation of acceleration when riding in an elevator and starting, stopping, or turning in an automobile. Accelerations produce dynamic forces on physical systems, as expressed in Newton's second law, F=ma. Force is proportional to acceleration, for a constant mass. The dynamic forces produced within the human body in response to acceleration can be harmful if excessive.

The human body is, after all, not rigid. It is a loosely coupled bag of water and tissue, most of which is quite internally mobile. Accelerations in the headward or footward directions will tend to either starve or flood the brain with blood as this liquid responds to Newton's law and effectively moves within the body in a direction opposite to the imposed acceleration as it lags the motion of the skeleton. Lack of blood supply to the brain causes blackout; excess blood supply causes redout. Either results in death if sustained for a long enough period.

A great deal of research has been done, largely by the military and NASA, to determine the limits of human tolerance to sustained accelerations in various directions. Figure 7-10 shows data developed from such tests. [1] The units of linear acceleration were defined in Table 1-4 (p. 19) as in/sec^2, ft/sec^2, or m/sec^2. Another common unit for acceleration is the *g*, defined as the acceleration due to gravity, which on Earth at sea level is approximately 386 in/sec^2, 32.2 ft/sec^2, or 9.8 m/sec^2. The *g* is a very convenient unit to use for accelerations involving the human as we live in a 1 *g* environment. Our weight, felt on our feet or buttocks, is defined by our mass times the acceleration due to gravity or *mg*. Thus an imposed acceleration of 1 g above the baseline of our gravity, or 2 *g's*, will be felt as a doubling of our weight. At 6 *g's* we would feel six times as heavy as normal and would have great difficulty even moving our arms against that acceleration. Figure 7-10 shows that the body's tolerance of acceleration is a function of its direction versus the body, its magnitude, and its duration. Note also that the data used for this chart were developed from tests on young, healthy military personnel in prime physical condition. The general population, children and elderly in particular, should not be expected to be able to withstand such high levels of acceleration. Since much machinery is designed for human use, these acceleration tolerance data should be of great interest and value to the machine designer. Several references dealing with these human factors data are provided in the bibliography to Chapter 1 (p. 20).

Another useful benchmark when designing machinery for human occupation is to attempt to relate the magnitudes of accelerations which you commonly experience to the calculated values for your potential design. Table 7-1 lists some approximate levels of acceleration, in *g's*, which humans can experience in everyday life. Your own experience of these will help you develop a "feel" for the values of acceleration which you encounter in designing machinery intended for human occupation.

Note that machinery which does not carry humans is limited in its acceleration levels only by considerations of the stresses in its parts. These stresses are often generated in large part by the dynamic forces due to accelerations. The range of acceleration values in such machinery is so wide that it is not possible to comprehensively define any guidelines for the designer as to acceptable or unacceptable levels of acceleration. If the moving mass is small, then very large numerical values of acceleration are reasonable. If the mass is large, the dynamic stresses which the materials can sustain may limit the allowable accelerations to low values. Unfortunately, the designer does not usually know how much acceleration is too much in a design until completing it to the point of calculating stresses in the parts. This usually requires a fairly complete and detailed design. If the stresses turn out to be too high and are due to dynamic forces, then the only recourse is to iterate back through the design process and reduce the accelerations and or masses in the design. This is one reason that the design process is a circular and not a linear one.



Average levels of linear acceleration, in different directions that can be tolerated on a voluntary basis for specified periods. Each curve shows the average *G* load that can be tolerated for the time indicated. The data points obtained were actually those on the axes; the lines as such are extrapolated from the data points to form the concentric figures.

(*Adapted from reference [1], Fig. 17-17, p. 505, reprinted with permission*)

FIGURE 7-10
Human tolerance of acceleration

As one point of reference, the acceleration of the piston in a small, four-cylinder economy car engine (about 1.5L displacement) at idle speed is about 40 *g's.* At highway speeds the piston acceleration may be as high as 700 *g's.* At the engine's top speed of 6000 rpm the peak piston acceleration is 2000 *g's!* As long as you're not riding on the piston, this is acceptable. These engines last a long time in spite of the high accelerations they experience. One key factor is the choice of low-mass, high-strength materials for the moving parts to both keep the dynamic forces down at these high accelerations and to enable them to tolerate high stresses.

7.7 JERK

No, not you! The **time derivative of acceleration** is called *jerk, pulse,* or *shock.* The name is apt, as it conjures the proper image of this phenomenon. **Jerk** is *the time rate of change of acceleration.* Force is proportional to acceleration. Rapidly changing acceleration means a rapidly changing force. Rapidly changing forces tend to "jerk" the object about! You have probably experienced this phenomenon when riding in an automobile. If the driver is inclined to 'jackrabbit" starts and accelerates violently away from the traffic light, you will suffer from large jerk because your acceleration will go from zero to a large value quite suddenly. But, when Jeeves, the chauffeur, is driving the *Rolls,* he always attempts to minimize jerk by accelerating gently and smoothly, so that *Madame* is entirely unaware of the change.

TABLE 7-1 Common Values of Acceleration in Human Activities

Gentle acceleration in an automobile	0.1 *g*
Jet aircraft on takeoff	0.3 *g*
Hard acceleration in an automobile	0.5 *g*
Panic stop in an automobile	0.7 *g*
Fast cornering in an automobile	0.8 *g*
Roller coaster	3.5 *g*
F-16 Air Force jet	9.0 *g*

Controlling and minimizing jerk in machine design is often of interest, especially if low vibration is desired. Large magnitudes of jerk will tend to excite the natural frequencies of vibration of the machine or structure to which it is attached and cause increased vibration and noise levels. Jerk control is of greater interest in the design of cams than of linkages, and we will investigate it in more detail in Chapter 8 on cam design.

The procedure for calculating the jerk in a linkage is a straightforward extension of the methods shown for acceleration analysis. Let angular jerk be represented by:

$$\varphi = \frac{d\alpha}{dt} \tag{7.33a}$$

and linear jerk by:

$$\mathbf{J} = \frac{d\mathbf{A}}{dt} \tag{7.33b}$$

To solve for jerk in a fourbar linkage, for example, the vector loop equation for acceleration (equation 7.7) is differentiated versus time. Refer to Figure 7-5 (p. 309) for notation.

$$\begin{aligned}
&-a\omega_2^3 je^{j\theta_2} - 2a\omega_2\alpha_2 e^{j\theta_2} + a\alpha_2\omega_2 j^2 e^{j\theta_2} + a\varphi_2 je^{j\theta_2}\\
&\quad - b\omega_3^3 je^{j\theta_3} - 2b\omega_3\alpha_3 e^{j\theta_3} + b\alpha_3\omega_3 j^2 e^{j\theta_3} + b\varphi_3 je^{j\theta_3}\\
&\qquad + c\omega_4^3 je^{j\theta_4} + 2c\omega_4\alpha_4 e^{j\theta_4} - c\alpha_4\omega_4 j^2 e^{j\theta_4} - c\varphi_4 je^{j\theta_4} = 0
\end{aligned} \tag{7.34a}$$

Collect terms and simplify:

$$\begin{aligned}
&-a\omega_2^3 je^{j\theta_2} - 3a\omega_2\alpha_2 e^{j\theta_2} + a\varphi_2 je^{j\theta_2}\\
&\quad - b\omega_3^3 je^{j\theta_3} - 3b\omega_3\alpha_3 e^{j\theta_3} + b\varphi_3 je^{j\theta_3}\\
&\qquad + c\omega_4^3 je^{j\theta_4} + 3c\omega_4\alpha_4 e^{j\theta_4} - c\varphi_4 je^{j\theta_4} = 0
\end{aligned} \tag{7.34b}$$

Substitute the Euler identity and separate into x and y components:

real part (x component):

$$\begin{aligned}
&a\omega_2^3 \sin\theta_2 - 3a\omega_2\alpha_2 \cos\theta_2 - a\varphi_2 \sin\theta_2\\
&\quad + b\omega_3^3 \sin\theta_3 - 3b\omega_3\alpha_3 \cos\theta_3 - b\varphi_3 \sin\theta_3\\
&\qquad - c\omega_4^3 \sin\theta_4 + 3c\omega_4\alpha_4 \cos\theta_4 + c\varphi_4 \sin\theta_4 = 0
\end{aligned} \tag{7.35a}$$

imaginary part (y component):

$$-a\omega_2^3 \cos\theta_2 - 3a\omega_2\alpha_2 \sin\theta_2 + a\varphi_2 \cos\theta_2$$
$$- b\omega_3^3 \cos\theta_3 - 3b\omega_3\alpha_3 \sin\theta_3 + b\varphi_3 \cos\theta_3$$
$$+ c\omega_4^3 \cos\theta_4 + 3c\omega_4\alpha_4 \sin\theta_4 - c\varphi_4 \cos\theta_4 = 0 \tag{7.35b}$$

These can be solved simultaneously for φ_3 and φ_4, which are the only unknowns. The driving angular jerk, φ_2, if nonzero, must be known in order to solve the system. All the other factors in equations 7.35 are defined or have been calculated from the position, velocity, and acceleration analyses. To simplify these expressions we will set the known terms to temporary constants.

In equation 7.35a, let:

$$\begin{array}{lll} A = a\omega_2^3 \sin\theta_2 & D = b\omega_3^3 \sin\theta_3 & G = 3c\omega_4\alpha_4 \cos\theta_4 \\ B = 3a\omega_2\alpha_2 \cos\theta_2 & E = 3b\omega_3\alpha_3 \cos\theta_3 & H = c\sin\theta_4 \\ C = a\varphi_2 \sin\theta_2 & F = c\omega_4^3 \sin\theta_4 & K = b\sin\theta_3 \end{array} \tag{7.36a}$$

Equation 7.35a then reduces to:

$$\varphi_3 = \frac{A - B - C + D - E - F + G + H\varphi_4}{K} \tag{7.36b}$$

Note that equation 7.36b defines angle φ_3 in terms of angle φ_4. We will now simplify equation 7.35b and substitute equation 7.36b into it.

In equation 7.35b, let:

$$\begin{array}{lll} L = a\omega_2^3 \cos\theta_2 & P = b\omega_3^3 \cos\theta_3 & S = c\omega_4^3 \cos\theta_4 \\ M = 3a\omega_2\alpha_2 \sin\theta_2 & Q = 3b\omega_3\alpha_3 \sin\theta_3 & T = 3c\omega_4\alpha_4 \sin\theta_4 \\ N = a\varphi_2 \cos\theta_2 & R = b\cos\theta_3 & U = c\cos\theta_4 \end{array} \tag{7.37a}$$

Equation 7.35b then reduces to:

$$R\varphi_3 - U\varphi_4 - L - M + N - P - Q + S + T = 0 \tag{7.37b}$$

Substituting equation 7.36b in equation 7.35b:

$$R\left(\frac{A - B - C + D - E - F + G + H\varphi_4}{K}\right) - U\varphi_4 - L - M + N - P - Q + S + T = 0 \tag{7.38}$$

The solution is:

$$\varphi_4 = \frac{KN - KL - KM - KP - KQ + AR - BR - CR + DR - ER - FR + GR + KS + KT}{KU - HR} \tag{7.39}$$

The result from equation 7.39 can be substituted into equation 7.36b to find φ_3. Once the angular jerk values are found, the linear jerk at the pin joints can be found from:

$$
\begin{aligned}
\mathbf{J}_A &= -a\omega_2^3 je^{j\theta_2} - 3a\omega_2\alpha_2 e^{j\theta_2} + a\varphi_2 je^{j\theta_2} \\
\mathbf{J}_{BA} &= -b\omega_3^3 je^{j\theta_3} - 3b\omega_3\alpha_3 e^{j\theta_3} + b\varphi_3 je^{j\theta_3} \\
\mathbf{J}_B &= -c\omega_4^3 je^{j\theta_4} - 3c\omega_4\alpha_4 e^{j\theta_4} + c\varphi_4 je^{j\theta_4} = 0
\end{aligned} \tag{7.40}
$$

The same approach as used in Section 7.4 (p. 319) to find the acceleration of any point on any link can be used to find the linear jerk at any point.

$$\mathbf{J}_P = \mathbf{J}_A + \mathbf{J}_{PA} \tag{7.41}$$

The jerk difference equation 7.41 can be applied to any point on any link if we let P represent any arbitrary point on any link and A represent any reference point on the same link for which we know the value of the jerk vector. Note that if you substitute equations 7.40 into 7.41, you will get equation 7.34.

7.8 LINKAGES OF *N* BARS

The same analysis techniques presented here for position, velocity, acceleration, and jerk, using the fourbar and fivebar linkage as the examples, can be extended to more complex assemblies of links. Multiple vector loop equations can be written around a linkage of arbitrary complexity. The resulting vector equations can be differentiated and solved simultaneously for the variables of interest. In some cases, the solution will require simultaneous solution of a set of nonlinear equations. A root-finding algorithm such as the Newton-Raphson method will be needed to solve these more complicated cases. A computer is necessary. An equation solver software package such as *TKSolver* or *Mathcad* that will do an iterative root-finding solution will be a useful aid to the solution of any of these analysis problems, including the examples shown here.

7.9 REFERENCES

1 Sanders, M. S., and E. J. McCormick, *Human Factors in Engineering and Design*, 6th ed., McGraw-Hill Co., New York, 1987, p. 505.

7.10 PROBLEMS

7-1 A point at a 6.5-in radius is on a body which is in pure rotation with $\omega = 100$ rad/sec and a constant $\alpha = -500$ rad/sec^2 at point A. The rotation center is at the origin of a coordinate system. When the point is at position A, its position vector makes a 45° angle with the X axis. It takes 0.01 sec to reach point B. Draw this system to some convenient scale, calculate the θ and ω of position B, and:

a. Write an expression for the particle's acceleration vector in position A using complex number notation, in both polar and cartesian forms.
b. Write an expression for the particle's acceleration vector in position B using complex number notation, in both polar and cartesian forms.
c. Write a vector equation for the acceleration difference between points B and A. Substitute the complex number notation for the vectors in this equation and solve for the acceleration difference numerically.
d. Check the result of part c with a graphical method.