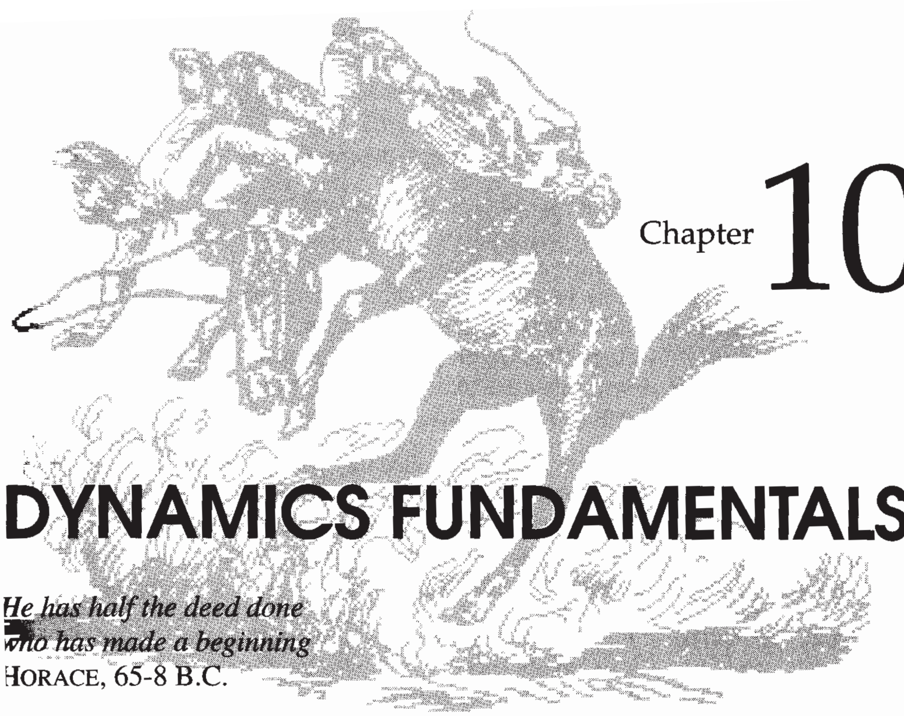


Chapter 10

DYNAMICS FUNDAMENTALS

He has half the deed done
who has made a beginning
HORACE, 65-8 B.C.

10.0 INTRODUCTION

Part I of this text has dealt with the **kinematics** of mechanisms while temporarily ignoring the forces present in those mechanisms. This second part will address the problem of determining the forces present in moving mechanisms and machinery. This topic is called **kinetics** or **dynamic force analysis**. We will start with a brief review of some fundamentals needed for dynamic analysis. It is assumed that the reader has had an introductory course in dynamics. If that topic is rusty, one can review it by referring to reference [1] or to any other text on the subject.

10.1 NEWTON'S LAWS OF MOTION

Dynamic force analysis involves the application of **Newton's** three **laws of motion** which are:

1 *A body at rest tends to remain at rest and a body in motion at constant velocity will tend to maintain that velocity unless acted upon by an external force.*

2 *The time rate of change of momentum of a body is equal to the magnitude of the applied force and acts in the direction of the force.*

3 *For every action force there is an equal and opposite reaction force.*

The second law is expressed in terms of rate of change of *momentum*, $\mathbf{M} = m\mathbf{v}$, where m is mass and $\mathbf{v}$ is velocity. Mass m is assumed to be constant in this analysis. The time rate of change of $m\mathbf{v}$ is $m\mathbf{a}$, where $\mathbf{a}$ is the acceleration of the mass center.

$$\mathbf{F} = m\mathbf{a} \tag{10.1}$$

$\mathbf{F}$ is the resultant of all forces on the system acting at the mass center.

We can differentiate between two subclasses of dynamics problems depending upon which quantities are known and which are to be found. The "**forward dynamics problem**" is the one in which we know everything about the external loads (forces and/or torques) being exerted on the system, and we wish to determine the accelerations, velocities, and displacements which result from the application of those forces and torques. This subclass is typical of the problems you probably encountered in an introductory dynamics course, such as determining the acceleration of a block sliding down a plane, acted upon by gravity. Given $\mathbf{F}$ and m, solve for $\mathbf{a}$.

The second subclass of dynamics problem, called the "**inverse dynamics problem**" is one in which we know the (desired) accelerations, velocities, and displacements to be imposed upon our system and wish to solve for the magnitudes and directions of the forces and torques which are necessary to provide the desired motions and which result from them. This inverse dynamics case is sometimes also called **kinetostatics**. Given $\mathbf{a}$ and m, solve for $\mathbf{F}$.

Whichever subclass of problem is addressed, it is important to realize that they are both dynamics problems. Each merely solves $\mathbf{F} = m\mathbf{a}$ for a different variable. To do so we must first review some fundamental geometric and mass properties which are needed for the calculations.

10.2 DYNAMIC MODELS

It is often convenient in dynamic analysis to create a simplified model of a complicated part. These models are sometimes considered to be a collection of point masses connected by massless rods. For a model of a rigid body to be dynamically equivalent to the original body, three things must be true:

1. *The mass of the model must equal that of the original body.*
2. *The center of gravity must be in the same location as that of the original body.*
3. *The mass moment of inertia must equal that of the original body.*

10.3 MASS

Mass is not weight! Mass is an invariant property of a rigid body. The weight of the same body varies depending on the gravitational system in which it sits. See Section 1.10 (p. 16) for a discussion of the use of proper mass units in various measuring systems. We will assume the mass of our parts to be constant in our calculations. For most earthbound machinery, this is a reasonable assumption. The rate at which an automobile or bulldozer loses mass due to fuel consumption, for example, is slow enough to be ignored when calculating dynamic forces over short time spans. However, this would not be a safe assumption for a vehicle such as the space shuttle, whose mass changes rapidly and drastically during liftoff.

When designing machinery, we must first do a complete kinematic analysis of our design, as described in Part I of this text, in order to obtain information about the accelerations of the moving parts. We next want to use Newton's second law to calculate the dynamic forces. But to do so we need to know the masses of all the moving parts which have these known accelerations. These parts do not exist yet! As with any design problem, we lack sufficient information at this stage of the design to accurately determine the best sizes and shapes of the parts. We must estimate the masses of the links and other parts of the design in order to make a first pass at the calculation. We will then have to iterate to better and better solutions as we generate more information. See Section 1.5 (p. 7) on the design process to review the use of iteration in design.

A first estimate of your parts' masses can be obtained by assuming some reasonable shapes and sizes for all the parts and choosing appropriate materials. Then calculate the volume of each part and multiply its volume by the material's mass density (not weight density) to obtain a first approximation of its mass. These mass values can then be used in Newton's equation. The densities of some common engineering materials can be found in Appendix B.

How will we know whether our chosen sizes and shapes of links are even acceptable, let alone optimal? Unfortunately, we will not know until we have carried the computations all the way through a complete stress and deflection analysis of the parts. It is often the case, especially with long, thin elements such as shafts or slender links, that the deflections of the parts under their dynamic loads will limit the design even at low stress levels. In some cases the stresses will be excessive.

We will probably discover that the parts fail under the dynamic forces. Then we will have to go back to our original assumptions about the shapes, sizes, and materials of these parts, redesign them, and repeat the force, stress, and deflection analyses. Design is, unavoidably, an iterative process.

The topic of stress and deflection analysis is beyond the scope of this text and will not be further discussed here. It is mentioned only to put our discussion of dynamic force analysis into context. We are analyzing these dynamic forces primarily to provide the information needed to do the stress and deflection analyses on our parts! It is also worth noting that, unlike a static force situation in which a failed design might be fixed by adding more mass to the part to strengthen it, to do so in a dynamic force situation can have a deleterious effect. More mass with the same acceleration will generate even higher forces and thus higher stresses! The machine designer often needs to remove mass (in the right places) from parts in order to reduce the stresses and deflections due to $F = ma$. Thus the designer needs to have a good understanding of both material properties and stress and deflection analysis to properly shape and size parts for minimum mass while maximizing the strength and stiffness needed to withstand the dynamic forces.

10.4 MASS MOMENT AND CENTER OF GRAVITY

When the mass of an object is distributed over some dimensions, it will possess a moment with respect to any axis of choice. Figure 10-1 shows a mass of general shape in an xyz axis system. A differential element of mass is also shown. The mass moment (first moment of mass) of the differential element is equal to the product of its mass and its distance from the axis of interest. With respect to the x, y, and z axes these are;

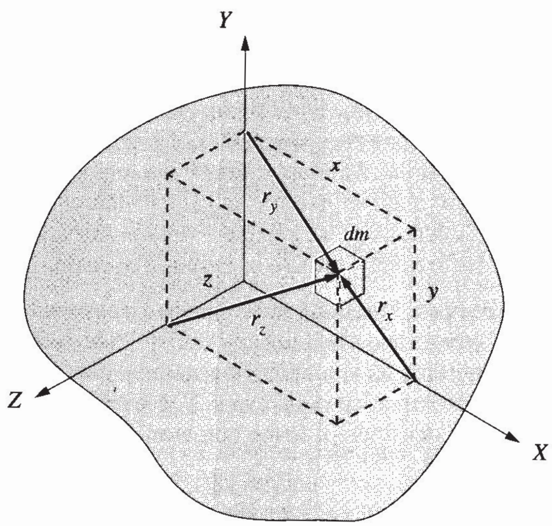


FIGURE 10-1
A generalized mass in a 3-D coordinate system

$$dM_x = r_x^1 dm = \sqrt{(y^2 + z^2)}\, dm \tag{10.2a}$$

$$dM_y = r_y^1 dm = \sqrt{(x^2 + z^2)}\, dm \tag{10.2b}$$

$$dM_z = r_z^1 dm = \sqrt{(x^2 + y^2)}\, dm \tag{10.2c}$$

The radius from the axis of interest to the differential element is shown with an exponent of 1 to emphasize the reason for this property being called the first moment of mass. To obtain the mass moment of the entire body we integrate each of these expressions.

$$M_x = \int \sqrt{(y^2 + z^2)}\, dm \tag{10.3a}$$

$$M_y = \int \sqrt{(x^2 + z^2)}\, dm \tag{10.3b}$$

$$M_z = \int \sqrt{(x^2 + y^2)}\, dm \tag{10.3c}$$

If the mass moment with respect to a particular axis is numerically zero, then that axis passes through the **center of mass (CM)** of the object, which for earthbound systems is coincident with its **center of gravity** *(CG)*. By definition the summation of first moments about all axes through the center of gravity is zero. We will need to locate the CG of all moving bodies in our designs because the linear acceleration component of each body is calculated acting at that point.

It is often convenient to model a complicated shape as several interconnected simple shapes whose individual geometries allow easy computation of their masses and the locations of their local *CGs*. The global *CG* can then be found from the summation of the first moments of these simple shapes set equal to zero. Appendix C contains formulas for the volumes and locations of centers of gravity of some common shapes.

Figure 10-2 shows a simple model of a mallet broken into two cylindrical parts, the handle and the head, which have masses m_h and m_d, respectively. The individual centers of gravity of the two parts are at l_d and $l_h/2$, respectively, with respect to the axis *ZZ*. We want to find the location of the composite center of gravity of the mallet with respect to *ZZ*. Summing the first moments of the individual components about *ZZ* and setting them equal to the moment of the entire mass about *ZZ*.

$$\sum M_{ZZ} = m_h \frac{l_h}{2} + m_d l_d = \left(m_h + m_d\right) d \tag{10.3d}$$

This equation can be solved for the distance *d* along the *X* axis, which, in this symmetrical example, is the only dimension of the composite *CG* not discernible by inspection. The *y* and *z* components of the composite *CG* are both zero.

$$d = \frac{m_h \frac{l_h}{2} + m_d l_d}{\left(m_h + m_d\right)} \tag{10.3e}$$

10.5 MASS MOMENT OF INERTIA (SECOND MOMENT OF MASS)

Newton's law applies to systems in rotation as well as to those in translation. The rotational form of Newton's second law is:

$$\mathbf{T} = I\alpha \tag{10.4}$$

where **T** is resultant torque about the mass center, α is angular acceleration, and *I* is mass moment of inertia about an axis through the mass center.

Mass moment of inertia is referred to some axis of rotation, usually one through the *CG*. Refer again to Figure 10-1 which shows a mass of general shape and an *xyz* axis system. A differential element of mass is also shown. The **mass moment of inertia** of the differential element is equal to the **product of its mass and the square of its distance** from the axis of interest. With respect to the *x, y,* and *z* axes they are:

$$dI_x = r_x^2 dm = (y^2 + z^2)dm \tag{10.5a}$$

$$dI_y = r_y^2 dm = (x^2 + z^2)dm \tag{10.5b}$$

$$dI_z = r_z^2 dm = (x^2 + y^2)dm \tag{10.5c}$$

The exponent of 2 on the radius term gives this property its other name of **second moment of mass.** To obtain the mass moments of inertia of the entire body we integrate each of these expressions.

$$I_x = \int (y^2 + z^2)dm \tag{10.6a}$$

$$I_y = \int (x^2 + z^2)dm \tag{10.6b}$$

$$I_z = \int (x^2 + y^2)dm \tag{10.6c}$$

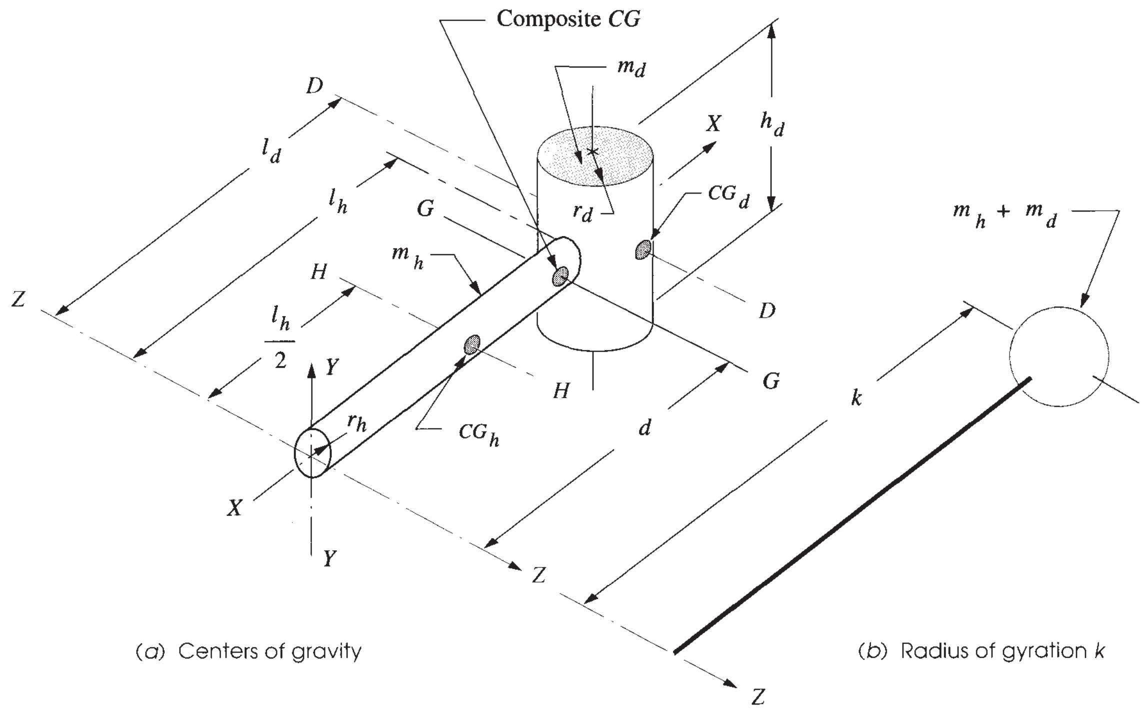


FIGURE 10-2

Dynamic models, composite center of gravity, and radius of gyration of a mallet

While it is fairly intuitive to appreciate the physical significance of the first moment of mass, it is more difficult to do the same for the second moment, or moment of inertia.

Consider equation 10.4. It says that torque is proportional to angular acceleration, and the constant of proportionality is this moment of inertia, *I*. Picture a common hammer or mallet as depicted in Figure 10-2. The head, made of steel, has large mass compared to the light wooden handle. When gripped properly, at the end of the handle, the radius to the mass of the head is large. Its contribution to the total *I* of the mallet is proportional to the square of the radius from the axis of rotation (your wrist at axis *ZZ*) to the head. Thus it takes considerably more torque to swing (and thus angularly accelerate) the mallet when it is held properly than if held near the head. As a child you probably chose to hold a hammer close to its head because you lacked the strength to provide the larger torque it needed when held properly. You also found it ineffective in driving nails when held close to the head because you were unable to store very much **kinetic energy** in it. In a translating system kinetic energy is:

$$KE = \frac{1}{2}mv^2 \tag{10.7a}$$

and in a rotating system kinetic energy is:

$$KE = \frac{1}{2}I\omega^2 \tag{10.7b}$$

Thus the kinetic energy stored in the mallet is also proportional to its moment of inertia I and to ω^2. So, you can see that holding the mallet close to its head reduces the I and lowers the energy available for driving the nail.

Moment of inertia then is one indicator of the ability of the body to store rotational kinetic energy and is also an indicator of the amount of torque that will be needed to rotationally accelerate the body. Unless you are designing a device intended for the storage and transfer of large amounts of energy (punch press, drop hammer, rock crusher etc.) you will probably be trying to minimize the moments of inertia of your rotating parts. Just as mass is a measure of resistance to linear acceleration, moment of inertia is a measure of resistance to angular acceleration. A large I will require a large driving torque and thus a larger and more powerful motor to obtain the same acceleration. Later we will see how to make moment of inertia work for us in rotating machinery by using flywheels with large I. The units of moment of inertia can be determined by doing a unit balance on either equation 10.4 or equation 10.7 and are shown in Table 1-4 (p. 19). In the **ips** system they are lb-in-sec^2 or blob-in^2. In the **SI** system, they are N-m-sec^2 or kg-m^2.

10.6 PARALLEL AXIS THEOREM (TRANSFER THEOREM)

The moment of inertia of a body with respect to any axis *(ZZ)* can be expressed as the sum of its moment of inertia about an axis *(GG)* parallel to *ZZ* through its *CG*, and the square of the perpendicular distance between those parallel axes.

$$I_{ZZ} = I_{GG} + md^2 \tag{10.8}$$

where ZZ and GG are parallel axes, *GG* goes through the *CG* of the body or assembly, m is the mass of the body or assembly, and d is the perpendicular distance between the parallel axes. This property is most useful when computing the moment of inertia of a complex shape which has been broken into a collection of simple shapes as shown in Figure 10-2a which represents a simplistic model of a mallet. The mallet is broken into two cylindrical parts, the handle and the head, which have masses m_h and m_d, and radii r_h and r_d, respectively. The expressions for the mass moments of inertia of a cylinder with respect to axes through its *CG* can be found in Appendix C and are for the handle about its *CG* axis *HH*:

$$I_{HH} = \frac{m_h\left(3r_h^2 + l_h^2\right)}{12} \tag{10.9a}$$

and for the head about its *CG* axis *DD*:

$$I_{DD} = \frac{m_d\left(3r_d^2 + h_d^2\right)}{12} \tag{10.9b}$$

Using the parallel axis theorem to transfer the moment of inertia to the axis *ZZ* at the end of the handle:

$$I_{ZZ} = \left[I_{HH} + m_h\left(\frac{l_h}{2}\right)^2\right] + \left[I_{DD} + m_d l_d{}^2\right] \tag{10.9c}$$

10.7 RADIUS OF GYRATION

The **radius of gyration** of a body is defined as the radius at which the entire mass of the body could be concentrated such that the resulting model will have the same moment of inertia as the original body. The mass of this model must be the same as that of the original body. Let I_{ZZ} represent the mass moment of inertia about *ZZ* from equation 10.9c and m the mass of the original body. From the parallel axis theorem, a concentrated mass m at a radius k will have a moment of inertia:

$$I_{ZZ} = mk^2 \tag{10.10a}$$

Since we want I_{ZZ} to be equal to the original moment of inertia, the required **radius of gyration** at which we will concentrate the mass m is then:

$$k = \sqrt{\frac{I_{ZZ}}{m}} \tag{10.10b}$$

Note that this property of radius of gyration allows the construction of an even simpler dynamic model of the system in which all the system mass is concentrated in a "point mass" at the end of a massless rod of length k. Figure 10-2b shows such a model of the mallet in Figure 10-2a.

By comparing equation 10.10a with equation 10.8, it can be seen that the radius of gyration k will always be larger than the radius to the composite *CG* of the original body.

$$I_{CG} + md^2 = I_{ZZ} = mk^2 \qquad \therefore k > d \tag{10.10c}$$

Appendix C contains formulas for the moments of inertia and radii of gyration of some common shapes.

10.8 CENTER OF PERCUSSION

The **center of percussion** is a point on a body which, when struck with a force, will have associated with it another point called the **center of rotation** at which there will be a zero reaction force. You have probably experienced the result of "missing the center of percussion" when you hit a baseball or softball with the wrong spot on the bat. The "right place on the bat" to hit the ball is the center of percussion associated with the point that your hands grip the bat (the center of rotation). Hitting the ball at other than the center of percussion results in a stinging force being delivered to your hands. Hit the right spot and you feel no force (nor pain). The center of percussion is sometimes called the "sweet spot" on a bat, tennis racquet, or golf club. In the case of our mallet example, a center of percussion at the head corresponds to a center of rotation near the end of the handle, and the handle is usually contoured to encourage gripping it there.

The explanation of this phenomenon is quite simple. To make the example two dimensional and eliminate the effects of friction, consider a hockey stick of mass m lying on the ice as shown in Figure 1O-3a. Strike it a sharp blow at point P with a force F perpendicular to the stick axis. The stick will begin to travel across the ice in complex planar motion, both rotating and translating. Its complex motion at any instant can be considered as the superposition of two components: pure translation of its center of gravity G in the direction ofF and pure rotation about that point G. Set up an embedded coordi-

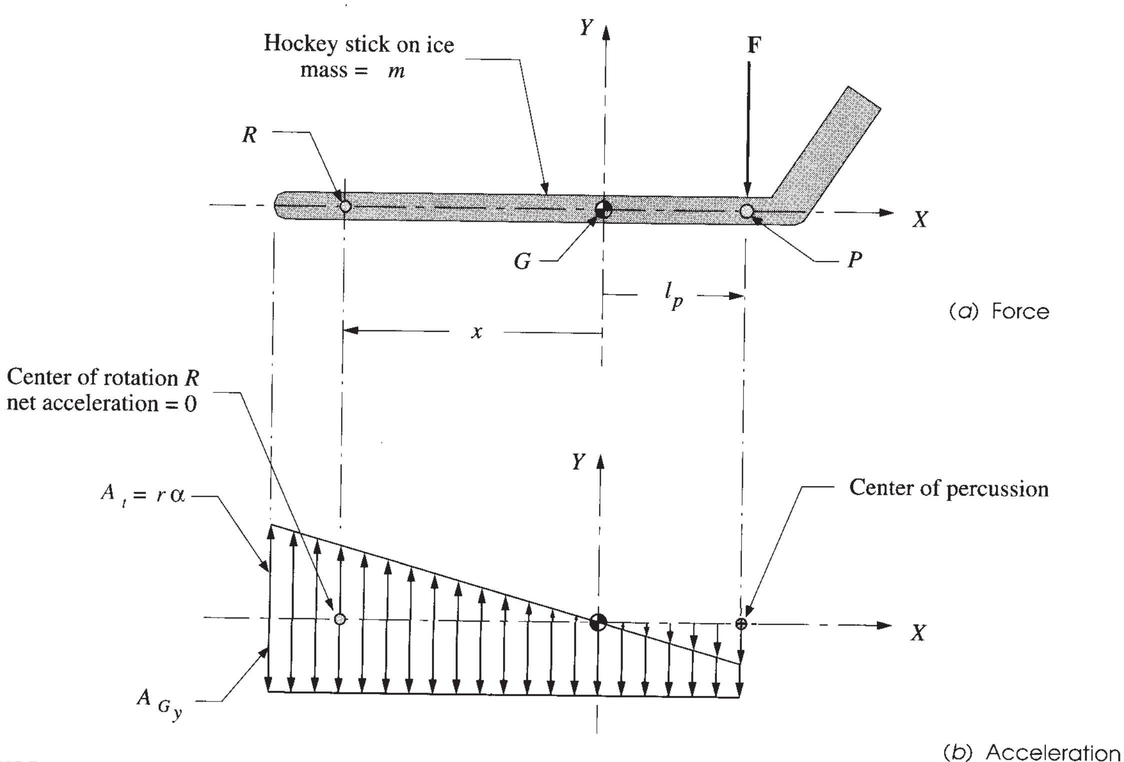


FIGURE 10-3

Center of percussion and center of rotation

nate system centered at G with the X axis along the stick in its initial position as shown. The translating component of acceleration of the CG resulting from the force **F** is (from Newton's law)

$$A_{G_y} = \frac{F}{m} \tag{10.11a}$$

and the angular acceleration is:

$$\alpha = \frac{T}{I_G} \tag{10.11b}$$

where I_G is its mass moment of inertia about the Z axis (out of the page) through the CG. But torque is also:

$$T = F l_P \tag{10.11c}$$

where l_p is the distance along the X axis from point G to point P so:

$$\alpha = \frac{F l_P}{I_G} \tag{10.11d}$$

The total linear acceleration at any point along the stick will be the sum of the linear acceleration A_{Gy} of the CG and the tangential component $(r\alpha)$ of the angular acceleration as shown in Figure 10-3b.

$$A_{y_{total}} = A_{G_y} + r\alpha$$
$$= \frac{F}{m} + x\left(\frac{F l_P}{I_G}\right) \tag{10.12}$$

where x is the distance to any point along the stick. Equation 10.12 can be set equal to zero and solved for the value of x for which the $r\alpha$ component exactly cancels the A_{Gy} component. This will be the **center of rotation** at which there is no translating acceleration, and thus no linear dynamic force. The solution for x when $A_{ytotal} = 0$ is:

$$x = -\frac{I_G}{m l_P} \tag{10.13a}$$

and substituting equation 10.10b:

$$x = -\frac{k^2}{l_P} \tag{10.13b}$$

where the radius of gyration k is calculated with respect to the axis ZZ through the CG.

Note that this relationship between the center of percussion and the center of rotation involves only geometry and mass properties. The magnitude of the applied force is irrelevant, but its location lp completely determines x. Thus there is **not** just one center of percussion on a body. Rather there will be pairs of points. For every point (center of percussion) at which a force is applied there will be a corresponding center of rotation at which the reaction force felt will be zero. This center of rotation need not fall within the physical length of the body however. Consider the value of x predicted by equation IO.13b if you strike the body at its CG.

10.9 LUMPED PARAMETER DYNAMIC MODELS

Figure IO-4a shows a simple plate or disk cam driving a spring-loaded, roller follower. This is a force-closed system which depends on the spring force to keep the cam and follower in contact at all times. Figure 1O-4b shows a lumped parameter model of this system in which all the mass which moves with the follower train is lumped together as m, all the springiness in the system is lumped within the **spring constant** k, and all the **damping** or resistance to movement is lumped together as a damper with coefficient c. The sources of mass which contribute to m are fairly obvious. The mass of the follower stem, the roller, its pivot pin, and any other hardware attached to the moving assembly all add together to create m. Figure 1O-4c shows the free-body diagram of the system acted upon by the cam force Fe, the spring force Fs, and the damping force Fd. There will of course also be the effects of mass times acceleration on the system.

Spring Constant

We have been assuming all links and parts to be rigid bodies in order to do the kinematic analyses, but to do a more accurate force analysis we need to recognize that these bodies

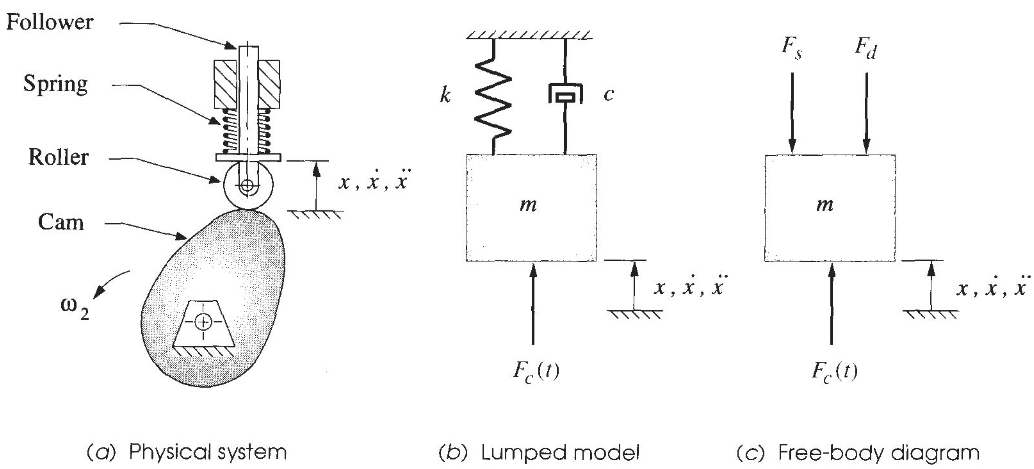


FIGURE 10-4

One-*DOF* lumped parameter model of a cam-follower system

are not truly rigid. The springiness in the system is assumed to be linear, thus describable by a spring constant k. A spring constant is defined as the force per unit deflection.

$$k = \frac{F_s}{x} \tag{10.14}$$

The total spring constant k of the system is a combination of the spring constants of the actual coil spring, plus the spring constants of all other parts which are deflected by the forces. The roller, its pin, and the follower stem are all springs themselves as they are made of elastic materials. The spring constant for any part can be obtained from the equation for its deflection under the applied loading. Any deflection equation relates force to displacement and can be algebraically rearranged to express a spring constant. An individual part may have more than one k if it is loaded in several modes as, for example, a camshaft with a spring constant in bending and also one in torsion. We will discuss the procedures for combining these various spring constants in the system together into a combined, effective spring constant k in the next section. For now let us just assume that we can so combine them for our analysis and create an overall k for our lumped parameter model.

Damping

The friction, more generally called **damping,** is the most difficult parameter of the three to model. It needs to be a combination of all the damping effects in the system. These may be of many forms. **Coulomb friction** results from two dry or lubricated surfaces rubbing together. The contact surfaces between cam and follower and between the follower and its sliding joint can experience coulomb friction. It is generally considered to be independent of velocity magnitude but has a different, larger value when velocity is zero (static friction force F_{st} or *stiction)* than when there is relative motion between the parts (dynamic friction F_d). Figure 1O-5a shows a plot of coulomb friction force versus relative velocity v at the contact surfaces. Note that friction always opposes motion, so

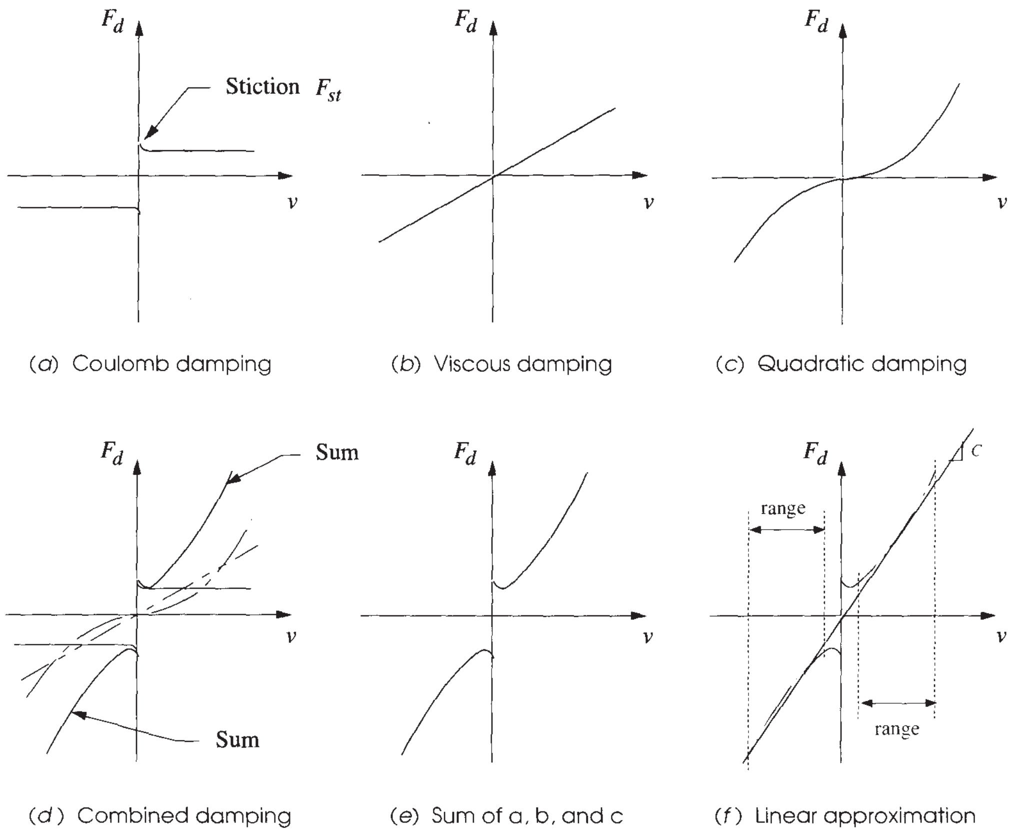


FIGURE 10-5
Modeling damping

the friction force abruptly changes sign at $v = 0$. The stiction F_{st} shows up as a larger spike at zero v than the dynamic friction value F_d. Thus, this is a **nonlinear** friction function. It is multivalued at zero. In fact, at zero velocity, the friction force can be any value between $-F_{st}$ and $+F_{st}$. It will be whatever force is needed to balance the system forces and create equilibrium. When the applied force exceeds F_{st}, motion begins and the friction force suddenly drops to F_d. This nonlinear damping creates difficulties in our simple model since we want to describe our system with linear differential equations having known solutions.

Other sources of damping may be present besides coulomb friction. **Viscous damping** results from the shearing of a fluid (lubricant) in the gap between the moving parts and is considered to be a linear function of relative velocity as shown in Figure 10-5b. **Quadratic damping** results from the movement of an object through a fluid medium as with an automobile pushing through the air or a boat through the water. This factor is a fairly negligible contributor to a cam-follower's overall damping unless the speeds are very high or the fluid medium very dense. Quadratic damping is a function of the square of the relative velocity as shown in Figure 10-5c. The relationship of the dynamic damping force F_d as a function of relative velocity for all these cases can be expressed as:

$$F_d = cv|v|^{r-1} \tag{10.15a}$$

where c is the constant damping coefficient, v is the relative velocity, and r is a constant which defines the type of damping.

For coulomb damping, $r = 0$ and:

$$F_d = \pm c \tag{10.15b}$$

For viscous damping, $r = 1$ and:

$$F_d = cv \tag{10.15c}$$

For quadratic damping, $r = 2$ and:

$$F_d = \pm cv^2 \tag{10.15d}$$

If we combine these three fonus of damping, their sum will look like Figure 1O-5d and e. This is obviously a nonlinear function. But we can approximate it over a reasonably small range of velocity as a linear function with a slope c which is then a *pseudo-viscous damping coefficient.* This is shown in Figure 1O-5f. While not an exact method to account for the true damping, this approach has been found to be acceptably accurate for a first approximation during the design process. The damping in these kinds of mechanical systems can vary quite widely from one design to the next due to different geometries, pressure or transmission angles, types of bearings, lubricants or their absence, etc. It is very difficult to accurately predict the level of damping (i.e., the value of c) in advance of the construction and testing of a prototype, which is the best way to determine the damping coefficient. If similar devices have been built and tested, their history can provide a good prediction. For the purpose of our dynamic modeling, we will assume *pseudo-viscous damping* and some value for c.

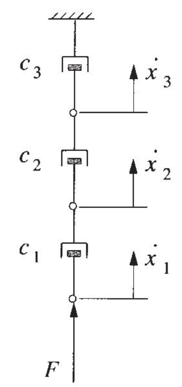
10.10 EQUIVALENT SYSTEMS

More complex systems than that shown in Figure 10-4 (p. 501) will have multiple masses, springs, and sources of damping connected together as shown in Figure 10-9. These models can be analyzed by writing dynamic equations for each subsystem and then solving the set of differential equations simultaneously. This allows a multi-degree-of-freedom analysis, with *one-DOF* for each subsystem included in the analysis. Koster[2] found in his extensive study of vibrations in cam mechanisms that a *five-DOF* model which included the effects of both torsional and bending deflection of the camshaft, backlash (see Section 10.2, p. 492) in the driving gears, squeeze effects of the lubricant, nonlinear coulomb damping, and motor speed variation gave a very good prediction of the actual, measured follower response. But he also found that a *single-DOF* model as shown in Figure 10-4 gave a reasonable simulation of the same system. We can then take the simpler approach and lump all the subsystems of Figure 10-9 together into a *single-DOF* **equivalent** system as shown in Figure 10-4. The combining of the various springs, dampers, and masses must be done carefully to properly approximate their dynamic interactions with each other.

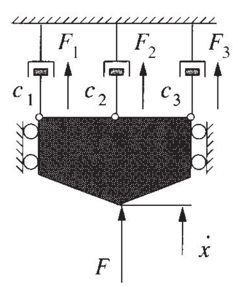
There are only two types of variables active in any dynamic system. These are given the general names of *through variable* and *across variable*. These names are descriptive of their actions within the system. A **through variable** *passes through the system*. An **across variable** *exists across the system*. The power in the system is the product of the through and across variables. Table 10-1 lists the through and across variables for various types of dynamic systems.

We commonly speak of the voltage across a circuit and the current flowing through it. We also can speak of the velocity across a mechanical "circuit" or system and the force which flows through it. Just as we can connect electrical elements such as resistors, capacitors, and inductors together in series or parallel or a combination of both to make an electrical circuit, we can connect their mechanical analogs, dampers, springs, and masses together in series, parallel, or a combination thereof to make a mechanical system. Table 10-2 shows the analogs between three types of physical systems. The fundamental relationships between through and across variables in electrical, mechanical, and fluid systems are shown in Table 10-3.

Recognizing a series or parallel connection between elements in an electrical circuit is fairly straightforward, as their interconnections are easily seen. Determining how mechanical elements in a system are interconnected is more difficult as their interconnections are sometimes hard to see. The test for series or parallel connection is best done by examining the forces and velocities (or the integral of velocity, displacement) that exist in the particular elements. If two elements have the same force passing through them, they are in series. If two elements have the same velocity or displacement, they are in parallel.



(*a*) Series



(*b*) Parallel

FIGURE 10-6

Dampers in series and in parallel

Combining Dampers

DAMPERS IN SERIES Figure 10-6a shows three dampers in series. The force passing through each damper is the same, and their individual displacements and velocities are different.

$$F = c_1(\dot{x}_1 - \dot{x}_2) = c_2(\dot{x}_2 - \dot{x}_3) = c_3\dot{x}_3$$

or:
$$\frac{F}{c_1} = \dot{x}_1 - \dot{x}_2; \qquad \frac{F}{c_2} = \dot{x}_2 - \dot{x}_3; \qquad \frac{F}{c_3} = \dot{x}_3$$

combining:
$$\dot{x}_{total} = (\dot{x}_1 - \dot{x}_2) + (\dot{x}_2 - \dot{x}_3) + \dot{x}_3 = \frac{F}{c_1} + \frac{F}{c_2} + \frac{F}{c_3}$$

then:
$$\dot{x}_{total} = F\frac{1}{c_{eff}} = F\left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right)$$

$$\frac{1}{c_{eff}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$$

$$c_{eff} = \frac{1}{\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}} \tag{10.16a}$$

The reciprocal of the effective damping of the dampers in series is the sum of the reciprocals of their individual damping coefficients.

TABLE 10-1 Through and Across Variables in Dynamic Systems

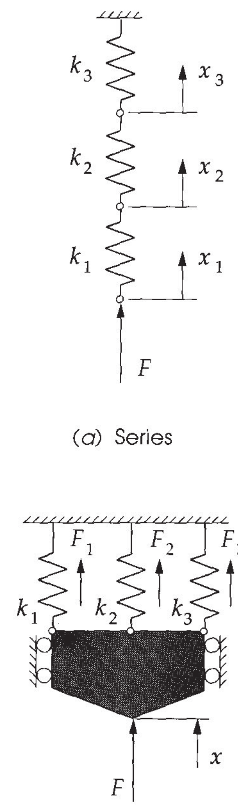
System Type	Through Variable	Across Variable	Power Units
Electrical	Current (i)	Voltage (e)	ei = watts
Mechanical	Force (F)	Velocity (v)	Fv = (in-lb)/sec
Fluid	Flow (Q)	Pressure (P)	PQ = (in-lb)/sec

TABLE 10-2 Physical Analogs in Dynamic Systems

System Type	Energy Dissipator	Energy Storage	Energy Storage
Electrical	Resistor (R)	Capacitor (C)	Inductor (L)
Mechanical	Damper (c)	Mass (m)	Spring (k)
Fluid	Fluid resistor (R_f)	Accumulator (C_f)	Fluid inductor (L_f)

TABLE 10-3 Relationships Between Variables in Dynamic Systems

System Type	Resistance	Capacitance	Inductance
Electrical	$i = \frac{1}{R} e$	$i = C \frac{de}{dt}$	$i = \frac{1}{L} \int e\, dt$
Mechanical	$F = c\, v$	$F = m \frac{dv}{dt}$	$F = k \int v\, dt$
Fluid	$Q = \frac{1}{R_f} P$	$Q = C_f \frac{dP}{dt}$	$Q = \frac{1}{L_f} \int P\, dt$



(a) Series

(b) Parallel

FIGURE 10-7
Springs in series and in parallel

Dampers in Parallel Figure 10-6b shows three dampers in parallel. The force passing through each damper is different, and their displacements and velocities are all the same.

$$F = F_1 + F_2 + F_3$$
$$F = c_1\dot{x} + c_2\dot{x} + c_3\dot{x}$$
$$F = (c_1 + c_2 + c_3)\dot{x}$$
$$F = c_{eff}\dot{x}$$
$$c_{eff} = c_1 + c_2 + c_3 \tag{10.16b}$$

The effective damping of the three is the sum of their individual damping coefficients.

Combining Springs

Springs are the mechanical analog of electrical inductors. Figure 10-7a shows three springs in series. The force passing through each spring is the same, and their individual displacements are different. A force F applied to the system will create a total deflection which is the sum of the individual deflections. The spring force is defined from the relationship in equation 10.14 (p. 501):

$$F = k_{eff} x_{total}$$

where:

$$x_{total} = (x_1 - x_2) + (x_2 - x_3) + x_3 \tag{10.17a}$$

$$(x_1 - x_2) = \frac{F}{k_1} \qquad (x_2 - x_3) = \frac{F}{k_2} \qquad x_3 = \frac{F}{k_3} \tag{10.17b}$$

Substituting, we find that the reciprocal of the effective k of **springs in series** is the sum of the reciprocals of their individual spring constants.

$$\frac{F}{k_{eff}} = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3}$$
$$k_{eff} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}} \tag{10.17c}$$

Figure 10-7b shows three springs in parallel. The force passing through each spring is different, and their displacements are all the same. The total force is the sum of the individual forces.

$$F_{total} = F_1 + F_2 + F_3 \tag{10.18a}$$

Substituting equation 10.17b we find that the effective k of **springs in parallel** is the sum of the individual spring constants.

$$k_{eff} x = k_1 x + k_2 x + k_3 x$$
$$k_{eff} = k_1 + k_2 + k_3 \tag{10.18b}$$

Combining Masses

Masses are the mechanical analog of electrical capacitors. The inertial forces associated with all moving masses are referenced to the ground plane of the system because the acceleration in $\mathbf{F} = m\mathbf{a}$ is absolute. Thus all masses are connected in parallel and combine in the same way as do capacitors in parallel with one terminal connected to a common ground.

$$m_{eff} = m_1 + m_2 + m_3 \tag{10.19}$$

Lever and Gear Ratios

Whenever an element is separated from the point of application of a force or from another element by a **lever ratio** or **gear ratio**, its effective value will be modified by that ratio. Figure 10-8a shows a spring at one end (A) and a mass at the other end (B) of a lever. We wish to model this system as a single-*DOF* lumped parameter system. There are two possibilities in this case. We can either transfer an equivalent mass m_{eff}

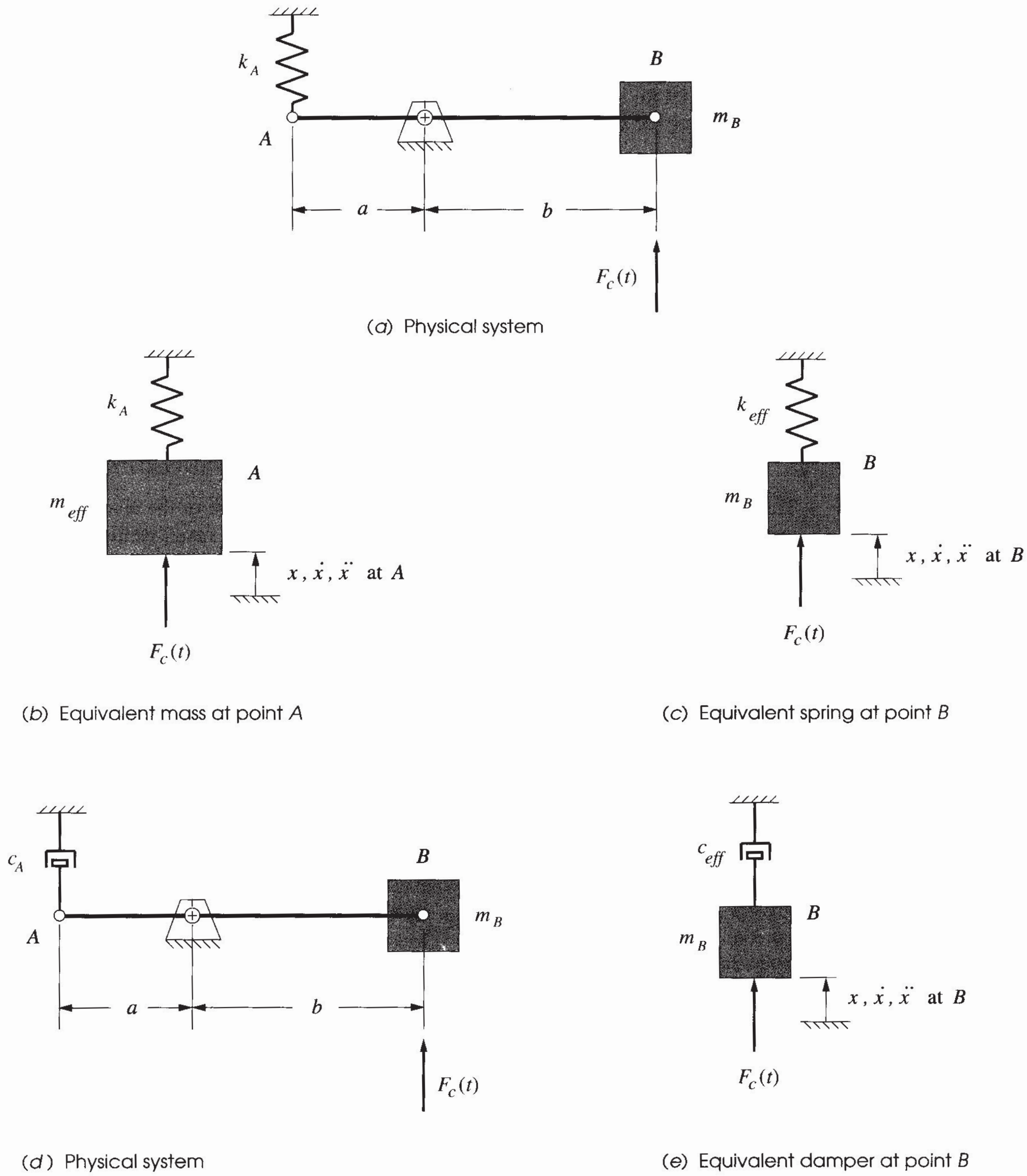


FIGURE 10-8

Lever or gear ratios affect the equivalent system to point A and attach it to the existing spring k, as shown in Figure 10-8b, or we can transfer an equivalent spring k_{eff} to point B and attach it to the existing mass m as shown in Figure 10-8c. In either case, for the lumped model to be equivalent to the original system, it must have the same energy in it.

Let's first find the effective mass that must be placed at point A to eliminate the lever. Equating the kinetic energies in the masses at points A and B:

$$\frac{1}{2} m_B v_B^2 = \frac{1}{2} m_{eff} v_A^2 \tag{10.20a}$$

The velocities at each end of the lever can be related by the lever ratio:

$$v_A = \left(\frac{a}{b}\right) v_B$$

substituting:

$$m_B v_B^2 = m_{eff} \left(\frac{a}{b}\right)^2 v_B^2$$

$$m_{eff} = \left(\frac{b}{a}\right)^2 m_B \tag{10.20b}$$

The effective mass varies from the original mass by the square of the lever ratio. Note that if the lever were instead a pair of gears of radii a and b, the result would be the same.

Now find the effective spring that would have to be placed at B to eliminate the lever. Equating the potential energies in the springs at points A and B:

$$\frac{1}{2} k_A x_A^2 = \frac{1}{2} k_{eff} x_B^2 \tag{10.21a}$$

The deflection at B is related to the deflection at A by the lever ratio:

$$x_B = \left(\frac{b}{a}\right) x_A$$

substituting:

$$k_A x_A^2 = k_{eff} \left(\frac{b}{a}\right)^2 x_A^2$$

$$k_{eff} = \left(\frac{a}{b}\right)^2 k_A \tag{10.21b}$$

The effective k varies from the original k by the square of the lever ratio. If the lever were instead a pair of gears of radii a and b, the result would be the same. So, gear or lever ratios can have a large effect on the lumped parameters' values in the simplified model.

Damping coefficients are also affected by the lever ratio. Figure 10-8d shows a damper and a mass at opposite ends of a lever. If the damper at A is to be replaced by a damper at B, then the two dampers must produce the same moment about the pivot, thus:

$$F_{d_A} a = F_{d_B} b \tag{10.21c}$$

Substitute the product of the damping coefficient and velocity for force:

$$\left(c_A \dot{x}_A\right)a = \left(c_{B_{eff}} \dot{x}_B\right)b \tag{10.21d}$$

The velocities at points A and B in Figure 10.8d can be related from kinematics:

$$\omega = \frac{\dot{x}_A}{a} = \frac{\dot{x}_B}{b}$$

$$\dot{x}_A = \dot{x}_B \frac{a}{b} \tag{10.21e}$$

Substituting in equation 10.21d we get an expression for the effective damping coefficient at B resulting from a damper at A.

$$\left(c_A \dot{x}_B \frac{a}{b}\right)a = \left(c_{B_{eff}} \dot{x}_B\right)b$$

$$c_{B_{eff}} = c_A \left(\frac{a}{b}\right)^2 \tag{10.21f}$$

Again, the square of the lever ratio determines the effective damping. The equivalent system is shown in Figure 10-8e.

EXAMPLE 10-1

Creating a Single-*DOF* Equivalent System Model of a Multielement Dynamic System.

Given: An automotive valve cam with translating flat follower, long pushrod, rocker arm, valve, and valve spring is shown in Figure 10-9a.

Problem: Create a suitable, approximate, single-*DOF*, lumped parameter model of the system. Define its effective mass, spring constant, and damping in terms of the individual elements' parameters.

Solution:

1 Break the system into individual elements as shown in Figure 10-9b. Each significant moving part is assigned a lumped mass element which has a connection to ground through a damper. There is also elasticity and damping within the individual elements, shown as connecting springs and dampers. The rocker arm is modeled as two lumped masses at its ends, connected with a rigid, massless rod for the crank and conrod of the slider-crank linkage. (See also Section 13.4, p. 614.) The breakdown shown represents a six-*DOF* model as there are six independent displacement coordinates, x_1 through x_6.

2 Define the individual spring constants of each element which represents the elasticity of a lumped mass from the elastic deflection formula for the particular part. For example, the pushrod is loaded in compression, so its relevant deflection formula and its k are,

$$x = \frac{Fl}{AE} \qquad \text{and} \qquad k_{pr} = \frac{F}{x} = \frac{AE}{l} \tag{a}$$

where A is the cross-sectional area of the pushrod, l is its length, and E is Young's modulus for the material. The k of the tappet element will have the same expression. The expression for

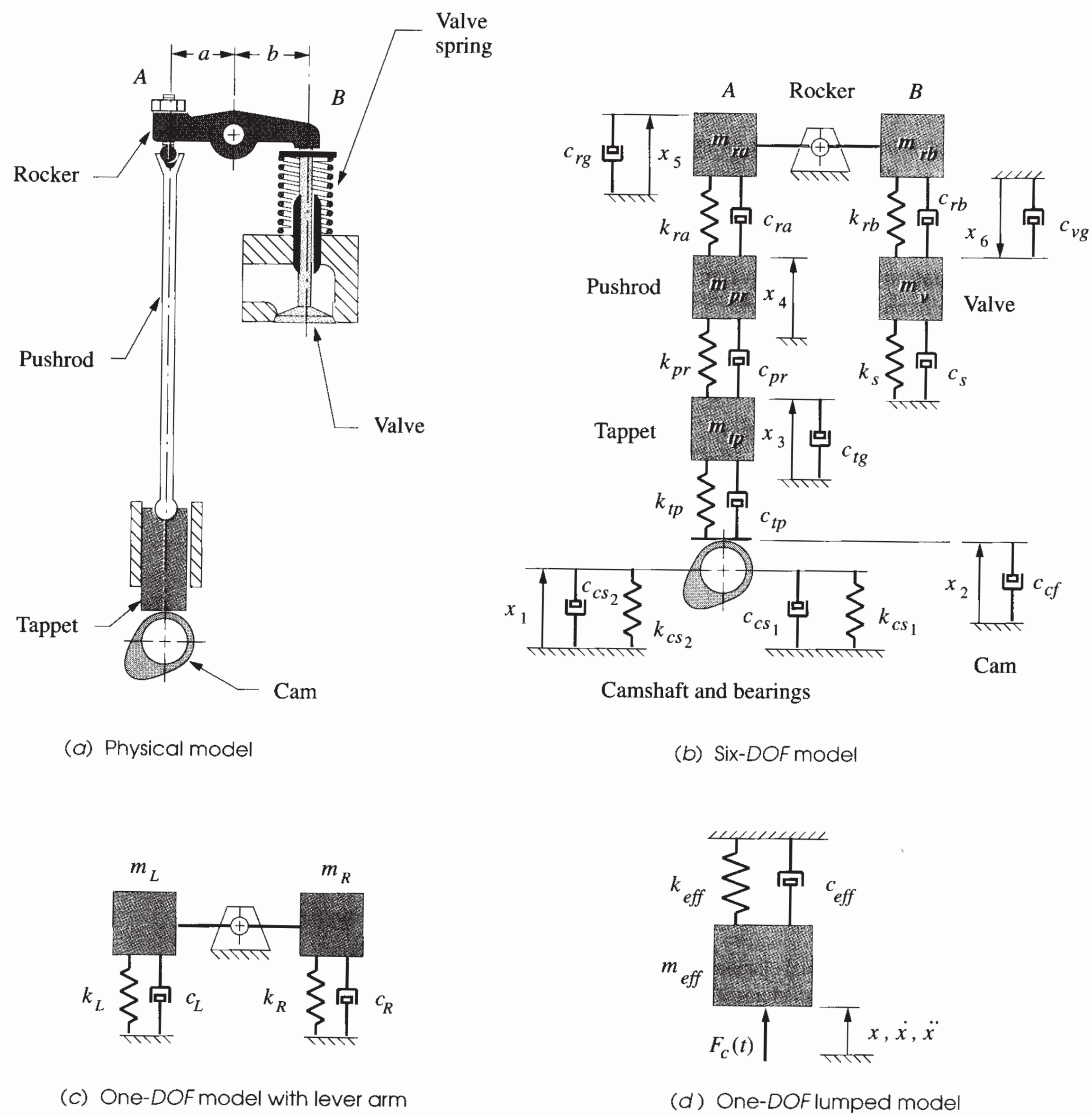


FIGURE 10-9

Lumped parameter models of an overhead valve engine cam-follower system

the k of a helical coil compression spring, as used for the valve spring, can be found in any spring design manual or machine design text and is:

$$k_{sp} = \frac{d^4 G}{8 D^3 N} \tag{b}$$

where d is the wire diameter, D is the mean coil diameter, N is the number of coils, and G is the modulus of rupture of the material.

The rocker arm also acts as a spring, as it is a beam in bending. It can be modeled as a double cantilever beam with its deflection on each side of the pivot considered separately. These spring effects are shown in the model as if they were compression springs, but that is just schematic. They really represent the bending deflection of the rocker arms. From the deflection formula for a cantilever beam with concentrated load:

$$x = \frac{Fl^3}{3EI} \qquad \text{and} \qquad k_{ra} = \frac{3EI}{l^3} \tag{c}$$

where I is the cross-sectional second moment of area of the beam, l is its length, and E is Young's modulus for the material. The spring constants of any other elements in a system can be obtained in similar fashion from their deflection formulas.

3 The dampers shown connected to ground represent the friction or viscous damping at the interfaces between the elements and the ground plane. The dampers between the masses represent the internal damping in the parts, which typically is quite small. These values will either have to be estimated from experience or measured in prototype assemblies.

4 The rocker arm provides a lever ratio which must be taken into account. The strategy will be to combine all elements on each side of the lever separately into two lumped parameter models as shown in Figure 10-9c, and then transfer one of those across the lever pivot to create one, single-*DOF* model as shown in Figure 10-9d.

5 The next step is to determine the types of connections, either series or parallel, between the elements. The masses are all in parallel as they each communicate their inertial force directly to ground and have independent displacements. On the left and right sides, respectively, the effective masses are:

$$m_L = m_{tp} + m_{pr} + m_{ra} \qquad\qquad m_R = m_{rb} + m_v \tag{d}$$

Note that m_v includes about one-third of the spring's mass. The two springs shown representing the bending deflection of the camshaft split the force between them, so they are in parallel and thus add directly.

$$k_{cs} = k_{cs_1} + k_{cs_2} \tag{e}$$

Note that, for completeness, the torsional deflection of the camshaft should also be included but is omitted in this example to reduce complexity. The combined camshaft spring constant and all the other springs shown on the left side are in series as they each have independent deflections and the same force passes through them all. The same is true of the springs on the right side. The effective spring constants for each side are then:

$$k_L = \frac{1}{\dfrac{1}{k_{cs}} + \dfrac{1}{k_{tp}} + \dfrac{1}{k_{pr}} + \dfrac{1}{k_{ra}}} \qquad\qquad k_R = \frac{1}{\dfrac{1}{k_{rb}} + \dfrac{1}{k_s}} \tag{f}$$

The dampers are in a combination of series and parallel. The pair of dampers c_{cs1} and c_{cs2} shown supporting the camshaft represent the friction in the two camshaft bearings and are in parallel.

$$c_{cs} = c_{cs_1} + c_{cs_2} \tag{g}$$

The ones representing internal damping are in series with one another and with the combined shaft damping.

$$c_{in_L} = \frac{1}{\dfrac{1}{c_{tp}} + \dfrac{1}{c_{pr}} + \dfrac{1}{c_{ra}} + \dfrac{1}{c_{cs}}} \qquad c_{in_R} = \frac{1}{\dfrac{1}{c_{rb}} + \dfrac{1}{c_s}} \tag{h}$$

where c_{inL} is all internal damping on the left side and c_{inR} is all internal damping on the right side of the rocker arm pivot. The combined internal damping c_{inL} goes to ground through c_{rg} and the combined internal damping c_{inR} goes to ground through the valve spring c_s. These two series combinations are then in parallel with all the other dampers that go to ground. The combined dampings for each side of the system are then:

$$c_L = c_{tg} + c_{rg} + c_{in_L} \qquad c_R = c_{vg} + c_{in_R} \tag{k}$$

6 The system can now be reduced to a single-*DOF* model with masses and springs lumped on either end of the rocker arm as shown in Figure 10-9c. We will bring the elements at point *B* across to point *A*. Note that we have reversed the sign convention across the pivot so that positive motion on one side results also in positive motion on the other. The damper, mass, and spring constant are affected by the square of the lever ratio as shown in equations 10.20 (p. 508) and 10.21 (p. 509).

$$\begin{aligned} m_{eff} &= m_L + \left(\frac{b}{a}\right)^2 m_R \\ k_{eff} &= k_L + \left(\frac{b}{a}\right)^2 k_R \\ c_{eff} &= c_L + \left(\frac{b}{a}\right)^2 c_R \end{aligned} \tag{m}$$

These are shown in Figure 10-9d on the final, one-*DOF* lumped model of the system.

Note that this one-*DOF* model provides only a relatively crude approximation of this complex system's behavior. Even though it may be an oversimplification, it is nevertheless still useful as a first approximation and serves in this context as an example of the general method involved in modelling dynamic systems. A more complex model with multiple degrees of freedom will provide a better approximation of the dynamic system's behavior.

10.11 SOLUTION METHODS

Dynamic force analysis can be done by any of several methods. Two will be discussed here, **superposition** and **linear simultaneous equation solution**. Both methods require that the system be linear.

These dynamic force problems typically have a large number of unknowns and thus have multiple equations to solve. The method of superposition attacks the problem by solving for parts of the solution and then adding (superposing) the partial results together to get the complete result. For example, if there are two loads applied to the system, we solve independently for the effects of each load, and then add the results. In effect we solve an N-variable system by doing sequential calculations on parts of the problem. It can be thought of as a "serial processing" approach.

Another method writes all the relevant equations for the entire system as a set of linear simultaneous equations. These equations can then be solved simultaneously to obtain the results. This can be thought of as analogous to a "parallel processing" approach. A convenient approach to the solution of sets of simultaneous equations is to put them in a standard matrix form and use a numerical matrix solver to obtain the answers. Matrix solvers are built into most engineering and scientific pocket calculators. Some spreadsheet packages and equation solvers will also do a matrix solution. A brief introduction to matrix solution of simultaneous equations was presented in Section 5.5. Appendix A describes the use of the computer program MATRIX, included on CD-ROM with this text. This program allows the rapid calculation of the solution to systems of up to 40 simultaneous equations. Please refer to the sections in Chapter 5 to review these calculation procedures and Appendix *A* for program MATRIX. Reference [3] provides an introduction to matrix algebra.

We will use both superposition and simultaneous equation solution to solve various dynamic force analysis problems in the remaining chapters. Both have their place, and one can serve as a check on the results from the other. So it is useful to be familiar with more than one approach. Historically, superposition was the only practical method for systems involving large numbers of equations until computers became available to solve large sets of simultaneous equations. Now the simultaneous equation solution method is more popular.

10.12 THE PRINCIPLE OF D'ALEMBERT

Newton's second law (equations 10.1, p. 492 and 10.4, p. 495) are all that are needed to solve any dynamic force system by the newtonian method. Jean le Rond d'Alembert (1717-1783), a French mathematician, rearranged Newton's equations to create a "quasi-static" situation from a dynamic one. D'Alembert's versions of equations 10.1 and 10.4 are:

$$\sum \mathbf{F} - m\mathbf{a} = 0$$
$$\sum \mathbf{T} - I\alpha = 0 \tag{10.22}$$

All d'Alembert did was to move the terms from the right side to the left, changing their algebraic signs in the process as required. These are obviously still the same equations as 10.1 and 10.4, algebraically rearranged. The motivation for this algebraic manipulation was to make the system look like a statics problem in which, for equilibrium, all forces and torques must sum to zero. Thus, this is sometimes called a quasi-static problem when expressed in this form. The premise is that by placing an "inertia force" equal to $-ma$ and an "inertia torque" equal to $-I\alpha$ on our free-body diagrams, the system will then be in a state of "dynamic equilibrium" and can be solved by the familiar methods of statics. These inertia forces and torques are equal in magnitude, opposite in sense, and along the same line of action as ma and $I\alpha$. This was a useful and popular approach which made the solution of dynamic force analysis problems somewhat easier when graphical vector solutions were the methods of choice.

With the availability, literally in your pocket or on your desk, of calculators and computers which can solve the simultaneous equations for these problems, there is now little motivation to labor through the complicated tedium of a graphical force analysis. It is for this reason that graphical force analysis methods are not presented in this text. However, d'Alembert's concept of "inertia forces and torques" still has, at a minimum, historical value and, in many instances, can prove useful in understanding what is going on in a dynamic system. Moreover, the concept of inertia force has entered the popular lexicon and is often used in a lay context when discussing motion. Thus we present a simple example of its use here and will use it again in our discussion of dynamic force analysis later in this text where it helps us to understand some topics such as balancing and superposition.

The popular term **centrifugal force**, used by laypersons everywhere to explain why a mass on a rope keeps the rope taut when swung in a circle, is in fact a d'Alembert inertial force. Figure 10-10a shows such a mass, being rotated at the end of a flexible but inextensible cord at a constant angular velocity ω and constant radius r. Figure 10-10b shows "pure" free-body diagrams of both members in this system, the ground link (1) and the rotating link (2). The only real force acting on link 2 is the force of link 1 on 2, F_{12}. Since angular acceleration is zero in this example, the acceleration acting on the link is only the $r\omega^2$ component, which is a **centripetal acceleration**, i.e., directed *toward the center*. The force at the pin from Newton's equation 10.1 is then:

$$F_{12} = mr\omega^2 \tag{10.23a}$$

Note that this force is directed toward the center, so it is a *centripetal* not a *centrifugal* (away from center) force. The force F_{21} which link 2 exerts on link 1 can be found from Newton's third law and is obviously equal and opposite to $\mathbf{F}_{12}$.

$$F_{21} = -F_{12} \tag{10.23b}$$

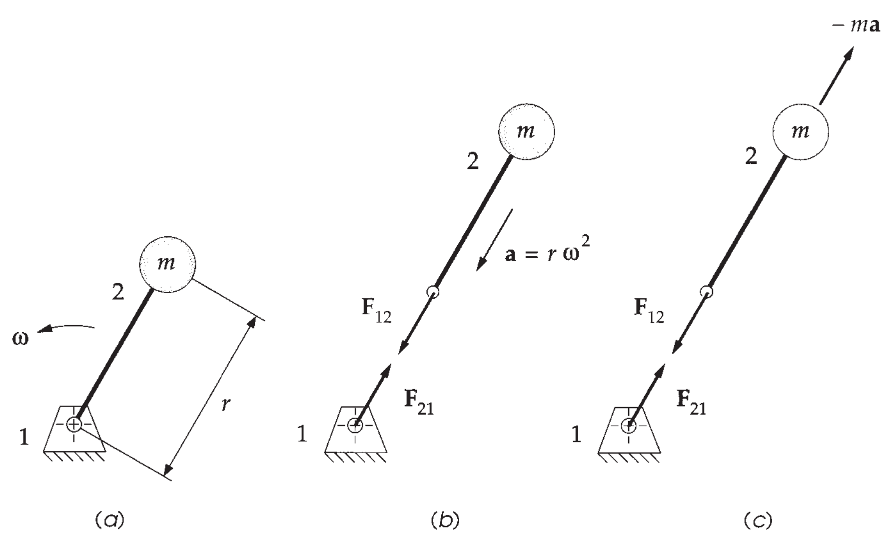


FIGURE 10-10
Centripetal and centrifugal forces

Thus it is the reaction force on link 1 which is centrifugal, not the force on link 2. Of course, it is this reaction force that your hand (link 1) feels, and this gives rise to the popular conception of something pulling centrifugally on the rotating weight. Now let us look at this through d'Alembert's eyes. Figure 10-10c shows another set of free-body diagrams done according to the principle of d'Alembert. Here we show a negative *ma* inertia force applied to the mass on link 2. The force at the pin from d'Alembert's equation is:

$$F_{12} - mr\omega^2 = 0$$

$$F_{12} = mr\omega^2 \tag{10.23c}$$

Not surprisingly, the result is the same as equation 10.23a, as it must be. The only difference is that the free-body diagram shows an inertia force applied to the rotating mass on link 2. This is the centrifugal force of popular repute which takes the blame for keeping the cord taut.

Clearly, any problem can be solved for the right answer no matter how we may algebraically rearrange the correct equations. So, if it helps our understanding to think in terms of these inertia forces being applied to a dynamic system, we will do so. When dealing with the topic of balancing, this approach does, in fact, help to visualize the effects of the balance masses on the system.

10.13 ENERGY METHODS—VIRTUAL WORK

The newtonian methods of dynamic force analysis described above have the advantage of providing complete information about all interior forces at pin joints as well as about the external forces and torques on the system. One consequence of this fact is the relative complexity of their application which requires the simultaneous solution of large systems of equations. Other methods are available for the solution of these problems which are easier to implement but give less information. Energy methods of solution are of this type. Only the external, work producing, forces and torques are found by these methods. The internal joint forces are not computed. One chief value of the energy approach is its use as a quick check on the correctness of the newtonian solution for input torque. Usually we are forced to use the more complete newtonian solution in order to obtain force information at pin joints so that pins and links can be analyzed for failure due to stress.

The *law of conservation of energy* states that energy can neither be created nor destroyed, only converted from one form to another. Most machines are designed specifically to convert energy from one form to another in some controlled fashion. Depending on the efficiency of the machine, some portion of the input energy will be converted to heat which cannot be completely recaptured. But large quantities of energy will typically be stored temporarily within the machine in both potential and kinetic form. It is not uncommon for the magnitude of this internally stored energy, on an instantaneous basis, to far exceed the magnitude of any useful external work being done by the machine.

Work is defined as *the dot product of force and displacement.* It can be positive, negative, or zero and is a scalar quantity.

$$W = \mathbf{F} \cdot \mathbf{R} \tag{10.24a}$$

Since the forces at the pin joints between the links have no relative displacement associated with them, they do no work on the system, and thus will not appear in the work equation. The work done by the system plus losses is equal to the energy delivered to the system.

$$E = W + Losses \tag{10.24b}$$

Pin-jointed linkages with low-friction bearings at the pivots can have high efficiencies, above 95%. Thus it is not unreasonable, for a first approximation in designing such a mechanism, to assume the losses to be zero. **Power** is the time rate of change of energy:

$$P = \frac{dE}{dt} \tag{10.24c}$$

Since we are assuming the machine member bodies to be rigid, only a change of position of the *CG*s of the members will alter the stored potential energy in the system. The gravitational forces of the members in moderate- to high-speed machinery often tend to be dwarfed by the dynamic forces from the accelerating masses. For these reasons we will ignore the weights and the gravitational potential energy and consider only the kinetic energy in the system for this analysis. The time rate of change of the kinetic energy stored within the system for linear and angular motion, respectively, is then:

$$\frac{d\left(\frac{1}{2} m\mathbf{v}^2\right)}{dt} = m\mathbf{a} \cdot \mathbf{v} \tag{10.25a}$$

and:

$$\frac{d\left(\frac{1}{2} I\omega^2\right)}{dt} = I\alpha \cdot \omega \tag{10.25b}$$

These are, of course, expressions for power in the system, equivalent to:

$$P = \mathbf{F} \cdot \mathbf{v} \tag{10.25c}$$

and:

$$P = \mathbf{T} \cdot \omega \tag{10.25d}$$

The rate of change of energy in the system at any instant must balance between that which is externally supplied and that which is stored within the system (neglecting losses). Equation 10.25a and b represent change in the energy stored in the system, and equation 10.25c and d represent change in energy passing into or out of the system. In the absence of losses, these two must be equal in order to conserve energy. We can express this relationship as a summation of all the delta energies (or power) due to each moving element (or link) in the system.

$$\sum_{k=2}^{n} \mathbf{F}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} \mathbf{T}_k \cdot \omega_k = \sum_{k=2}^{n} m_k \mathbf{a}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} I_k \alpha_k \cdot \omega_k \tag{10.26a}$$

The subscript k represents each of the n links or moving elements in the system, starting with link 2 because link 1 is the stationary ground link. Note that all the angular and linear velocities and accelerations in this equation must have been calculated, for all positions of the mechanism of interest, from a prior kinematic analysis. Likewise, the masses and mass moments of inertia of all moving links must be known.

If we use the principle of d'Alembert to rearrange this equation, we can more easily "name" the terms for discussion purposes.

$$\sum_{k=2}^{n} \mathbf{F}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} \mathbf{T}_k \cdot \omega_k - \sum_{k=2}^{n} m_k \mathbf{a}_k \cdot \mathbf{v}_k - \sum_{k=2}^{n} I_k \alpha_k \cdot \omega_k = 0 \tag{10.26b}$$

The first two terms in equation 10.26b represent, respectively, the change in energy due to all **external forces** and all **external torques** applied to the system. These would include any forces or torques from other mechanisms which impinge upon any of these links and also includes the driving torque. The second two terms represent, respectively, the change in energy due to all **inertia forces** and all **inertia torques** present in the system. These last two terms define the change in stored kinetic energy in the system at each time step. The only unknown in this equation when properly set up is the **driving torque** (or driving force) to be supplied by the mechanism's motor or actuator. This driving torque (or force) is then the only variable which can be solved for with this approach. The internal joint forces are not present in the equation as they do no net work on the system.

Equation 10.26b is sometimes called the **virtual work equation**, which is something of a misnomer, as it is in fact a **power equation**. When this analysis approach is applied to a statics problem, there is no motion. The term **virtual work** comes from the concept of each force causing an infinitesimal, or virtual, displacement of the static system element to which it is applied over an infinitesimal delta time. The dot product of the force and the virtual displacement is the virtual work. In the limit, this becomes the instantaneous power in the system. We will present an example of the use of this method of virtual work in the next chapter along with examples of the newtonian solution applied to linkages in motion.

10.14 REFERENCES

1 **Beer, F. P., and E. R. Johnson.** (1984). *Vector Mechanics for Engineers, Statics and Dynamics*, McGraw-Hill Inc., New York.

2 **Koster, M. P.** (1974). *Vibrations of Cam Mechanisms.* Phillips Technical Library Series, Macmillan: London.

3 **Jennings, A.** (1977). *Matrix Computation for Engineers and Scientists*, John Wiley and Sons, New York.