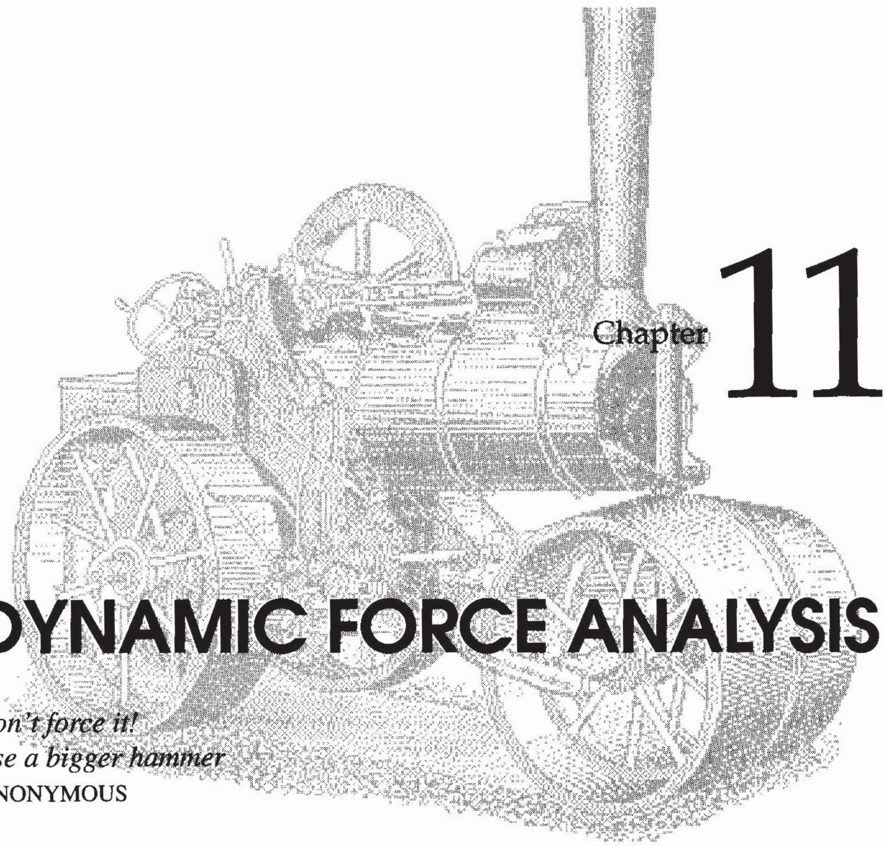
Chapter 11



DYNAMIC FORCE ANALYSIS

Don't force it!
Use a bigger hammer
ANONYMOUS

11.0 INTRODUCTION

When kinematic synthesis and analysis have been used to define a geometry and set of motions for a particular design task, it is logical and convenient to then use a **kinetostatic, or inverse dynamics,** solution to determine the forces and torques in the system. We will take that approach in this chapter and concentrate on solving for the forces and torques that result from, and are required to drive, our kinematic system in such a way as to provide the designed accelerations. Numerical examples are presented throughout this chapter. These examples are also provided as disk files for input to either program MATRIX or FOURBAR. These programs are described in Appendix A. The reader is encouraged to open the referenced files in these programs and investigate the examples in more detail. The file names are noted in the discussion of each example.

11.1 NEWTONIAN SOLUTION METHOD

Dynamic force analysis can be done by any of several methods. The one which gives the most information about forces internal to the mechanism requires only the use of Newton's law as defined in equations 10.1 (p. 492) and *IOA* (p. 495). These can be written as a summation of all forces and torques in the system.

$$\sum \mathbf{F} = m\mathbf{a} \qquad \sum \mathbf{T} = I_G \alpha \tag{11.1a}$$

It is also convenient to separately sum force components in X and Y directions, with the coordinate system chosen for convenience. The torques in our two dimensional system are all in the Z direction. This lets us break the two vector equations into three scalar equations:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum T = I_G \alpha \tag{11.1b}$$

These three equations must be written for each moving body in the system which will lead to a set of linear simultaneous equations for any system. The set of simultaneous equations can most conveniently be solved by a matrix method as was shown in Chapter 5. These equations do not account for the gravitational force (weight) on a link. If the kinematic accelerations are large compared to gravity, which is often the case, then the weight forces can be ignored in the dynamic analysis. If the machine members are very massive or moving slowly with small kinematic accelerations, or both, the weight of the members may need to be included in the analysis. The weight can be treated as an external force acting on the CG of the member at a constant angle.

11.2 SINGLE LINK IN PURE ROTATION

As a simple example of this solution procedure, consider the single link in pure rotation shown in Figure 11-la. In any of these kinetostatic dynamic force analysis problems, the kinematics of the problem must first be fully defined. That is, the angular accelerations of all rotating members and the linear accelerations of the *CGs* of all moving members must be found for all positions of interest. The mass of each member and the mass moment of inertia *10* with respect to each member's CG must also be known. In addition there may be external forces or torques applied to any member of the system. These are all shown in the figure.

While this analysis can be approached in many ways, it is useful for the sake of consistency to adopt a particular arrangement of coordinate systems and stick with it. We present such an approach here which, if carefully followed, will tend to minimize the chances of error. The reader may wish to develop his or her own approach once the principles are understood. The underlying mathematics is invariant, and one can choose coordinate systems for convenience. The vectors which are acting on the dynamic system in any loading situation are the same at a particular time regardless of how we may decide to resolve them into components for the sake of computation. The solution result will be the same.

We will first set up a nonrotating, local coordinate system on each moving member, located at its *CG.* (In this simple example we have only one moving member.) All externally applied forces, whether due to other connected members or to other systems must then have their points of application located in this local coordinate system. Figure 11-1b shows a free-body diagram of the moving link 2. The pin joint at 02 on link 2 has a force F 12 due to the mating link I, the x and y components of which are F_{12x} and F_{12y}' These subscripts are read "force of link I on 2" in the x or y direction. This subscript notation scheme will be used consistently to indicate which of the "action-reaction" pair of forces at each joint is being solved for.

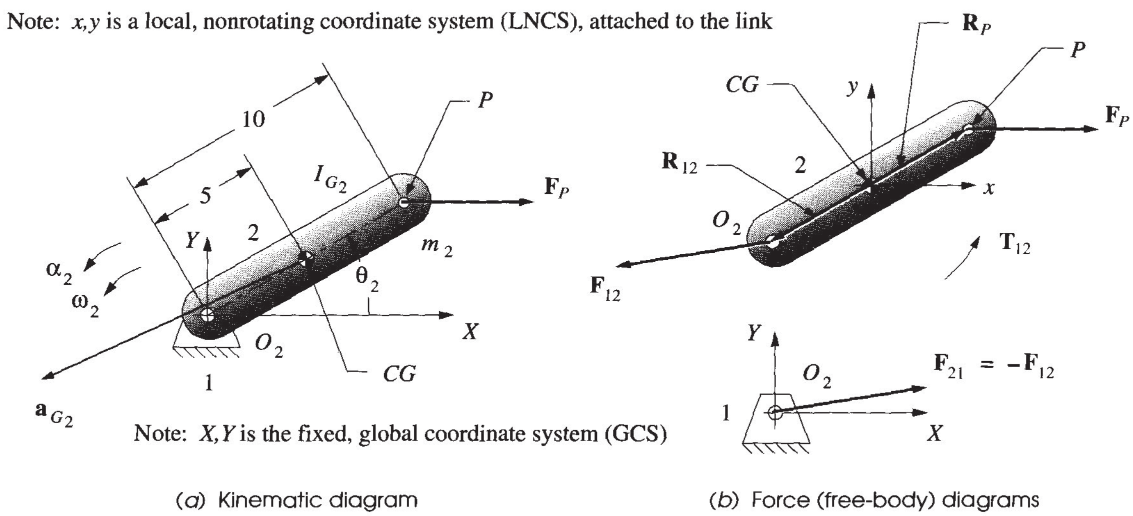


FIGURE 11-1

Dynamic force analysis of a single link in pure rotation

There is also an externally applied force $\mathbf{F}_P$ shown at point P, with components F_{Px} and F_{Py}. The points of application of these forces are defined by position vectors $\mathbf{R}_{12}$ and $\mathbf{R}_P$, respectively. These position vectors are defined with respect to the local coordinate system at the *CG* of the member. We will need to resolve them into x and y components. There will have to be a source torque available on the link to drive it at the kinematically defined accelerations. This is one of the unknowns to be solved for. The source torque is the torque delivered *from the ground to the driver link 2* and so is labeled $\mathbf{T}_{12}$. The other two unknowns in this example are the force components at the pin joint F_{12x} and F_{12y}.

We have three unknowns and three equations, so the system can be solved. Equations 11.1 can now be written for the moving link 2. Any applied forces or torques whose directions are known must retain the proper signs on their components. We will assume all unknown forces and torques to be positive. Their true signs will "come out in the wash."

$$\sum \mathbf{F} = \mathbf{F}_P + \mathbf{F}_{12} = m_2 \mathbf{a}_G$$
$$\sum \mathbf{T} = \mathbf{T}_{12} + (\mathbf{R}_{12} \times \mathbf{F}_{12}) + (\mathbf{R}_P \times \mathbf{F}_P) = I_G \alpha \tag{11.2}$$

The force equation can be broken into its two components. The torque equation contains two cross product terms which represent torques due to the forces applied at a distance from the *CG*. When these cross products are expanded, the system of equations becomes:

$$F_{P_x} + F_{12_x} = m_2 a_{G_x}$$
$$F_{P_y} + F_{12_y} = m_2 a_{G_y} \tag{11.3}$$
$$T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) = I_G \alpha$$

This can be put in matrix form with the coefficients of the unknown variables forming the **A** matrix, the unknown variables the **B** vector, and the constant terms the **C** vector and then solved for **B**.

$$[\mathbf{A}] \times [\mathbf{B}] = [\mathbf{C}]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12_y} & R_{12_x} & 1 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_x} - F_{P_x} \\ m_2 a_{G_y} - F_{P_y} \\ I_G \alpha - \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) \end{bmatrix} \tag{11.4}$$

Note that the **A** matrix contains all the geometric information and the **C** matrix contains all the dynamic information about the system. The **B** matrix contains all the unknown forces and torques. We will now present a numerical example to reinforce your understanding of this method.

EXAMPLE 11-1

Dynamic Force Analysis of a Single Link in Pure Rotation. (See Figure 11-1)

Given: The 10-in-long link shown weighs 4 lb. Its *CG* is on the line of centers at the 5-in point. Its mass moment of inertia about its *CG* is 0.08 lb-in-sec². Its kinematic data are:

θ_2 deg	ω_2 rad/sec	α_2 rad/sec^2	a_{G_2} in/sec^2
30	20	15	2001 @ 208°

An external force of 40 lb at 0° is applied at point *P*.

Find: The **force $\mathbf{F}_{12}$** at pin joint O_2 and the driving **torque $\mathbf{T}_{12}$** needed to maintain motion with the given acceleration for this instantaneous position of the link.

Solution:

1 Convert the given weight to proper mass units, in this case blobs:

$$mass = \frac{weight}{g} = \frac{4 \text{ lb}}{386 \text{ in/sec}^2} = 0.0104 \text{ blobs} \tag{a}$$

2 Set up a local coordinate system at the *CG* of the link and draw all applicable vectors acting on the system as shown in the figure. Draw a free-body diagram as shown.

3 Calculate the *x* and *y* components of the position vectors $\mathbf{R}_{12}$ and $\mathbf{R}_P$ in this coordinate system:

$$\mathbf{R}_{12} = 5\ in\ @\angle 210°; \qquad R_{12_x} = -4.33, \qquad R_{12_y} = -2.50$$
$$\mathbf{R}_P = 5\ in\ @\angle 30°; \qquad R_{P_x} = +4.33, \qquad R_{P_y} = +2.50 \qquad (b)$$

4 Calculate the x and y components of the acceleration of the CG in this coordinate system:

$$\mathbf{a}_G = 2001\ @\angle 208°; \qquad a_{G_x} = -1766.78, \qquad a_{G_y} = -939.41 \qquad (c)$$

5 Calculate the x and y components of the external force at P in this coordinate system:

$$\mathbf{F}_P = 40\ @\angle 0°; \qquad F_{P_x} = 40, \qquad F_{P_y} = 0 \qquad (d)$$

6 Substitute these given and calculated values into the matrix equation 11.4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2.50 & -4.33 & 1 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} (0.01)(-1766.78) - 40 \\ (0.01)(-939.41) - 0 \\ (0.08)(15) - \{(4.33)(0) - (2.5)(40)\} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2.50 & -4.33 & 1 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -57.67 \\ -9.39 \\ 101.2 \end{bmatrix} \qquad (e)$$

7 Solve this system either by inverting matrix **A** and premultiplying that inverse times matrix **C** using a pocket calculator such as the HP-15c or by inputting the values for matrices **A** and **C** to program MATRIX provided with this text.

Program MATRIX gives the following solution:

$$F_{12_x} = -57.67\text{ lb}, \qquad F_{12_y} = -9.39\text{ lb}, \qquad T_{12} = 204.72\text{ lb-in} \qquad (f)$$

Converting the force to polar coordinates:

$$\mathbf{F}_{12} = 58.43\ @\angle 189.25° \qquad (g)$$

Read the disk file E011-01.mat into program MATRIX to exercise this example.

11.3 FORCE ANALYSIS OF A THREEBAR CRANK-SLIDE LINKAGE

When there is more than one link in the assembly, the solution simply requires that the three equations 11.1b be written for each link and then solved simultaneously. Figure 11-2a shows a threebar crank-slide linkage. This linkage has been simplified from the fourbar slider-crank (see Figure 11-4) by replacing the kinematically redundant slider block (link 4) with a half joint as shown. This linkage transformation reduces the number of links to three with no change in degree of freedom (see Section 2.9, p. 40). Only links 2 and 3 are moving. Link 1 is ground. Thus we should expect to have six equations in six unknowns (three per moving link).

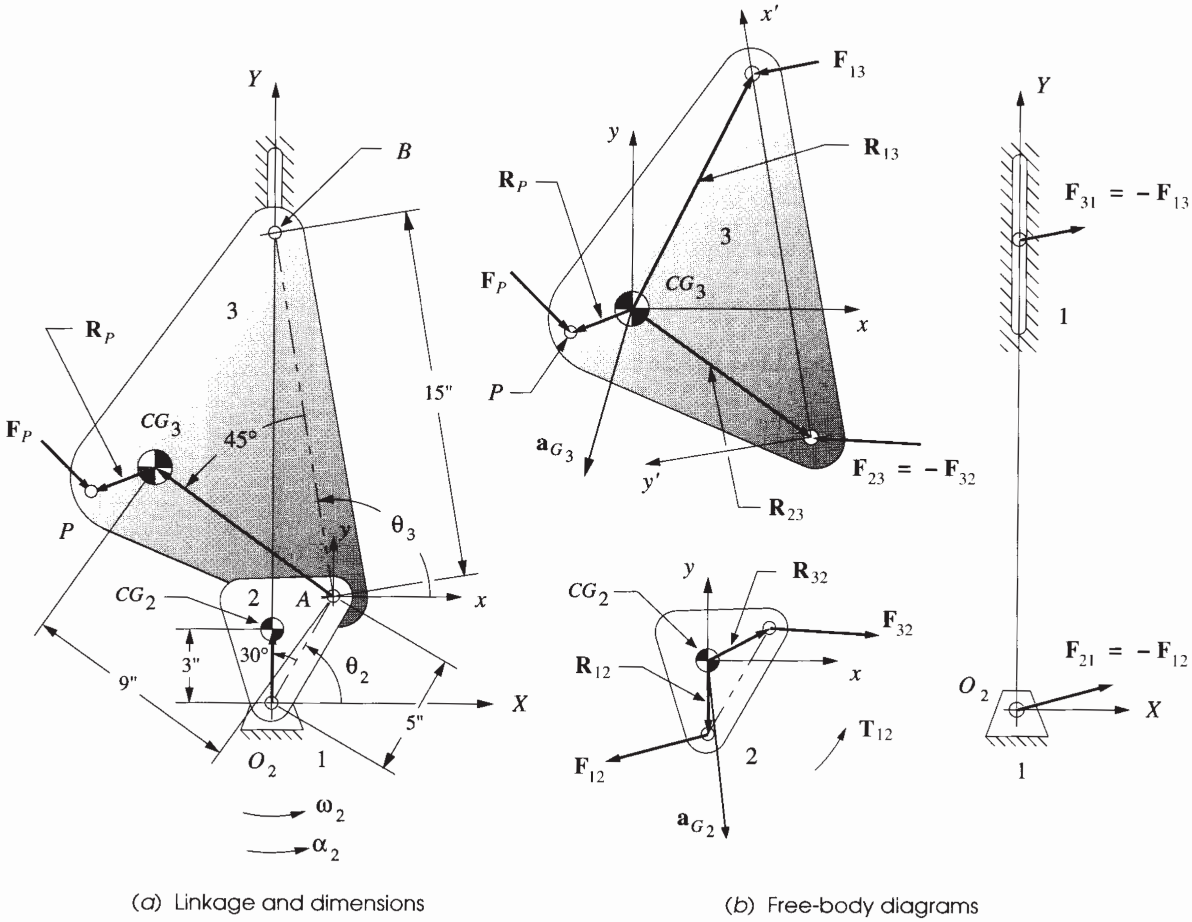


FIGURE 11-2
Dynamic force analysis of a slider-crank linkage

Figure 11-2b shows the linkage "exploded" into its three separate links, drawn as free bodies. A kinematic analysis must have been done in advance of this dynamic force analysis in order to determine, for each moving link, its angular acceleration and the linear acceleration of its *CG*. For the kinematic analysis, only the link lengths from pin to pin were required. For a dynamic analysis the mass (m) of each link, the location of its *CG*, and its mass moment of inertia (I_G) about that *CG* are also needed.

The *CG* of each link is initially defined by a position vector rooted at one pin joint whose angle is measured with respect to the line of centers of the link in the local, rotating coordinate system (LRCS) x', y'. This is the most convenient way to establish the *CG* location since the link line of centers is the kinematic definition of the link. However, we will need to define the link's dynamic parameters and force locations with respect to a local, nonrotating coordinate system (LNCS) x, y located at its *CG* and which is always parallel to the global coordinate system (GCS) XY. The position vector locations of all attachment points of other links and points of application of external forces must be defined with respect to the link's LNCS. Note that these kinematic and applied force data must be available for all positions of the linkage for which a force analysis is desired. In the following discussion and examples, only one linkage position will be addressed. The process is identical for each succeeding position and only the calculations must be repeated. Obviously, a computer will be a valuable aid in accomplishing the task.

Link 2 in Figure 11-2b shows forces acting on it at each pin joint, designated $\mathbf{F}_{12}$ and $\mathbf{F}_{32}$. By convention their subscripts denote the force that the adjoining link is exerting *on* the link being analyzed; that is, $\mathbf{F}_{12}$ is the force of 1 *on* 2 and $\mathbf{F}_{32}$ is the force of 3 *on* 2. Obviously there is also an equal and opposite force at each of these pins which would be designated as $\mathbf{F}_{21}$ and $\mathbf{F}_{23}$, respectively. The choice of which of the members of these pairs of forces to be solved for is arbitrary. As long as proper bookkeeping is done, their identities will be maintained.

When we move to link 3, we maintain the same convention of showing forces acting *on* the link in its free-body diagram. Thus at instant center I_{23} we show $\mathbf{F}_{23}$ acting on link 3. However, because we showed force $\mathbf{F}_{32}$ acting at the same point on link 2, this introduces an additional unknown to the problem for which we need an additional equation. The equation is available from Newton's third law:

$$\mathbf{F}_{23} = -\mathbf{F}_{32} \tag{11.5}$$

Thus we are free to substitute the negative reaction force for any action force at any joint. This has been done on link 3 in the figure in order to reduce the unknown forces at that joint to one, namely $\mathbf{F}_{32}$. The same procedure is followed at each joint with one of the action-reaction forces arbitrarily chosen to be solved for and its negative reaction applied to the mating link.

The naming convention used for the position vectors ($\mathbf{R}_{ap}$) which locate the pin joints with respect to the CG in the link's nonrotating local coordinate system is as follows. The first subscript (a) denotes the adjoining link to which the position vector points. The second subscript (p) denotes the parent link to which the position vector belongs. Thus in the case of link 2 in Figure 11-2b, vector $\mathbf{R}_{12}$ locates the attachment point of link 1 to link 2, and $\mathbf{R}_{32}$ the attachment point of link 3 to link 2. Note that in some cases these subscripts will match those of the pin forces shown acting at those points, but where the negative reaction force has been substituted as described above, the subscript order of the force and its position vector will not agree. This can lead to confusion and must be carefully watched for typographical errors when setting up the problem.

Any external forces acting on the links are located in similar fashion with a position vector to a point on the line of application of the force. This point is given the same letter subscript as that of the external force. Link 3 in the figure shows such an external force $\mathbf{F}_P$ acting on it at point P. The position vector $\mathbf{R}_P$ locates that point with respect to the CG. It is important to note that the CG of each link is consistently taken as the point of reference for all forces acting on that link. Left to its own devices, an unconstrained body in complex motion will spin about its own CG; thus we analyze its linear acceleration at that point and apply the angular acceleration about the CG as a center.

Equations 11.1 are now written for each moving link. For link 2, with the cross products expanded:

$$
\begin{aligned}
F_{12_x} + F_{32_x} &= m_2 a_{G_{2x}} \\
F_{12_y} + F_{32_y} &= m_2 a_{G_{2y}} \\
T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{32_x} F_{32_y} - R_{32_y} F_{32_x}\right) &= I_{G_2} \alpha_2
\end{aligned}
\tag{11.6a}
$$

For link 3, with the cross products expanded, note the substitution of the reaction force $-\mathbf{F}_{32}$ for $\mathbf{F}_{23}$:

$$
\begin{aligned}
F_{13_x} - F_{32_x} + F_{P_x} &= m_3 a_{G_{3x}} \\
F_{13_y} - F_{32_y} + F_{P_y} &= m_3 a_{G_{3y}} \\
\left(R_{13_x} F_{13_y} - R_{13_y} F_{13_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) &= I_{G_3} \alpha_3
\end{aligned}
\tag{11.6b}
$$

Note also that $\mathbf{T}_{12}$, the source torque, only appears in the equation for link 2 as that is the driver crank to which the motor is attached. Link 3 has no externally applied torque but does have an external force $\mathbf{F}_P$ which might be due to whatever link 3 is pushing on to do its external work.

There are seven unknowns present in these six equations, F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{13x}, F_{13y}, and T_{12}. But, F_{13y} is due only to friction at the joint between link 3 and link 1. We can write a relation for the friction force f at that interface such as $f = \pm\mu N$, where $\pm\mu$ is a known coefficient of coulomb friction. The friction force always opposes motion. The kinematic analysis will provide the velocity of the link at the sliding joint. The direction of f will always be the opposite of this velocity. Note that μ is a nonlinear function which has a discontinuity at zero velocity; thus at the linkage positions where velocity is zero, the inclusion of μ in these linear equations is not valid. (See Figure 10-5a, p. 502.) In this example, the normal force N is equal to F_{13x} and the friction force f is equal to F_{13y}. For linkage positions with nonzero velocity, we can eliminate F_{13y} by substituting into equation 11.6b,

$$
F_{13_y} = \mu\, F_{13_x} \tag{11.6c}
$$

where the sign of F_{13y} is taken as the opposite of the sign of the velocity at that point. We are then left with six unknowns in equations 11.6 and can solve them simultaneously. We also rearrange equations 11.6a and 11.6b to put all known terms on the right side.

$$
\begin{aligned}
F_{12_x} + F_{32_x} &= m_2 a_{G_{2x}} \\
F_{12_y} + F_{32_y} &= m_2 a_{G_{2y}} \\
T_{12} + R_{12_x} F_{12_y} - R_{12_y} F_{12_x} + R_{32_x} F_{32_y} - R_{32_y} F_{32_x} &= I_{G_2} \alpha_2 \\
F_{13_x} - F_{32_x} &= m_3 a_{G_{3_x}} - F_{P_x} \\
\pm\mu\, F_{13_x} - F_{32_y} &= m_3 a_{G_{3_y}} - F_{P_y} \\
\left(\pm\mu\, R_{13_x} - R_{13_y}\right) F_{13_x} - R_{23_x} F_{32_y} + R_{23_y} F_{32_x} &= I_{G_3} \alpha_3 - R_{P_x} F_{P_y} + R_{P_y} F_{P_x}
\end{aligned}
\tag{11.6d}
$$

Putting these six equations in matrix form we get:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & \mu & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & \left(\mu R_{13_x} - R_{13_y}\right) & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{13_x} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2}\alpha_2 \\ m_3 a_{G_{3x}} - F_{P_x} \\ m_3 a_{G_{3y}} - F_{P_y} \\ I_{G_3}\alpha_3 - R_{P_x}F_{P_y} + R_{P_y}F_{P_x} \end{bmatrix} \tag{11.7}$$

This system can be solved by using program MATRIX or any other matrix solving calculator. As an example of this solution consider the following linkage data.

EXAMPLE 11-2

Dynamic Force Analysis of a Threebar Crank-Slide Linkage with Half Joint. (See Figure 11-2, p. 526.)

Given: The 5-in long crank (link 2) shown weighs 2 lb. Its *CG* is at 3 in and 30° from the line of centers. Its mass moment of inertia about its *CG* is 0.05 lb-in-sec^2. Its acceleration is defined in its LNCS, *x,y*. Its kinematic data are:

θ_2 deg	ω_2 rad/sec	α_2 rad/sec^2	a_{G_2} in/sec^2
60	30	–10	2700.17 @ –89.4°

The coupler (link 3) is 15 in long and weighs 4 lb. Its *CG* is at 9 in and 45° from the line of centers. Its mass moment of inertia about its *CG* is 0.10 lb-in-sec^2. Its acceleration is defined in its LNCS, *x,y*. Its kinematic data are:

θ_3 deg	ω_3 rad/sec	α_3 rad/sec^2	a_{G_3} in/sec^2
99.59	–8.78	–136.16	3453.35 @ 254.4°

The sliding joint on link 3 has a velocity of 96.95 in/sec in the +*Y* direction.

There is an external force of 50 lb at –45°, applied at point *P* which is located at 2.7 in and 101° from the *CG* of link 3, measured in the link's embedded, rotating coordinate system or LRCS *x'*, *y'* (origin at *A* and *x* axis from *A* to *B*). The coefficient of friction μ is 0.2.

Find: The **forces** $\mathbf{F}_{12}$, $\mathbf{F}_{32}$, $\mathbf{F}_{13}$ at the joints and the driving **torque** $\mathbf{T}_{12}$ needed to maintain motion with the given acceleration for this instantaneous position of the link.

Solution:

1 Convert the given weights to proper mass units, in this case blobs:

$$mass_{link2} = \frac{weight}{g} = \frac{2 \text{ lb}}{386 \text{ in/sec}^2} = 0.0052 \text{ blobs} \qquad (a)$$

$$mass_{link3} = \frac{weight}{g} = \frac{4 \text{ lb}}{386 \text{ in/sec}^2} = 0.0104 \text{ blobs} \qquad (b)$$

2 Set up a local, nonrotating *xy* coordinate system (LNCS) at the *CG* of each link, and draw all applicable position and force vectors acting within or on that system as shown in Figure 11-2. Draw a free-body diagram of each moving link as shown.

3 Calculate the *x* and *y* components of the position vectors $\mathbf{R}_{12}$, $\mathbf{R}_{32}$, $\mathbf{R}_{23}$, $\mathbf{R}_{13}$, and $\mathbf{R}_P$ in the LNCS coordinate system:

$$\begin{array}{lll}
\mathbf{R}_{12} = 3.00 \ @ \angle\ 270.0°; & R_{12_x} = 0, & R_{12_y} = -3.0 \\
\mathbf{R}_{32} = 2.83 \ @ \angle\ 28.0°; & R_{32_x} = 2.500, & R_{32_y} = 1.333 \\
\mathbf{R}_{23} = 9.00 \ @ \angle\ 324.5°; & R_{23_x} = 7.329, & R_{23_y} = -5.224 \\
\mathbf{R}_{13} = 10.72 \ @ \angle\ 63.14°; & R_{13_x} = 4.843, & R_{13_y} = 9.563 \\
\mathbf{R}_{P} = 2.70 \ @ \angle\ 201.0°; & R_{P_x} = -2.521, & R_{P_y} = -0.968
\end{array} \qquad (c)$$

These position vector angles are measured with respect to the LNCS which is always parallel to the global coordinate system (GCS), making the angles the same in both systems.

4 Calculate the *x* and *y* components of the acceleration of the *CG*s of all moving links in the global coordinate system:

$$\begin{array}{lll}
\mathbf{a}_{G_2} = 2700.17 \ @ \angle\ -89.4°; & a_{G_{2x}} = 28.28, & a_{G_{2y}} = -2700 \\
\mathbf{a}_{G_3} = 3453.35 \ @ \angle\ 254.4°; & a_{G_{3x}} = -930.82, & a_{G_{3y}} = -3325.54
\end{array} \qquad (d)$$

5 Calculate the *x* and *y* components of the external force at *P* in the global coordinate system:

$$\mathbf{F}_P = 50 \ @ \angle -45°; \qquad F_{P_x} = 35.36, \qquad F_{P_y} = -35.36 \qquad (e)$$

6 Substitute these given and calculated values into the matrix equation 11.7.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1.333 & 2.5 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0.2 & 0 \\ 0 & 0 & -5.224 & -7.329 & [(0.2)4.843-(9.563)] & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{13_x} \\ T_{12} \end{bmatrix} =$$

$$\begin{bmatrix} (0.005)(28.28) \\ (0.005)(-2700) \\ (0.05)(-10) \\ (0.01)(-930.82)-35.36 \\ (0.01)(-3325.54)-(-35.36) \\ (0.1)(-136.16)-(-2.521)(-35.36)+(-0.968)(35.36) \end{bmatrix} = \begin{bmatrix} 0.141 \\ -13.500 \\ -.500 \\ -44.668 \\ 2.105 \\ -136.987 \end{bmatrix} \tag{f}$$

7 Solve this system either by inverting matrix **A** and premultiplying that inverse times matrix **C** using a pocket calculator such as the HP-15c, or by inputting the values for matrices **A** and **C** to program MATRIX provided with this text which gives the following solution:

$$\begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{13_x} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -39.232 \\ -10.336 \\ 39.373 \\ -3.164 \\ -5.295 \\ 177.590 \end{bmatrix} \tag{g}$$

Converting the forces to polar coordinates:

$$\begin{aligned} \mathbf{F}_{12} &= 40.57 \text{ lb} \quad @ \angle\ 194.76° \\ \mathbf{F}_{32} &= 39.50 \text{ lb} \quad @ \angle\ -4.60° \\ \mathbf{F}_{13} &= 5.40 \text{ lb} \quad @ \angle\ 191.31° \end{aligned} \tag{h}$$

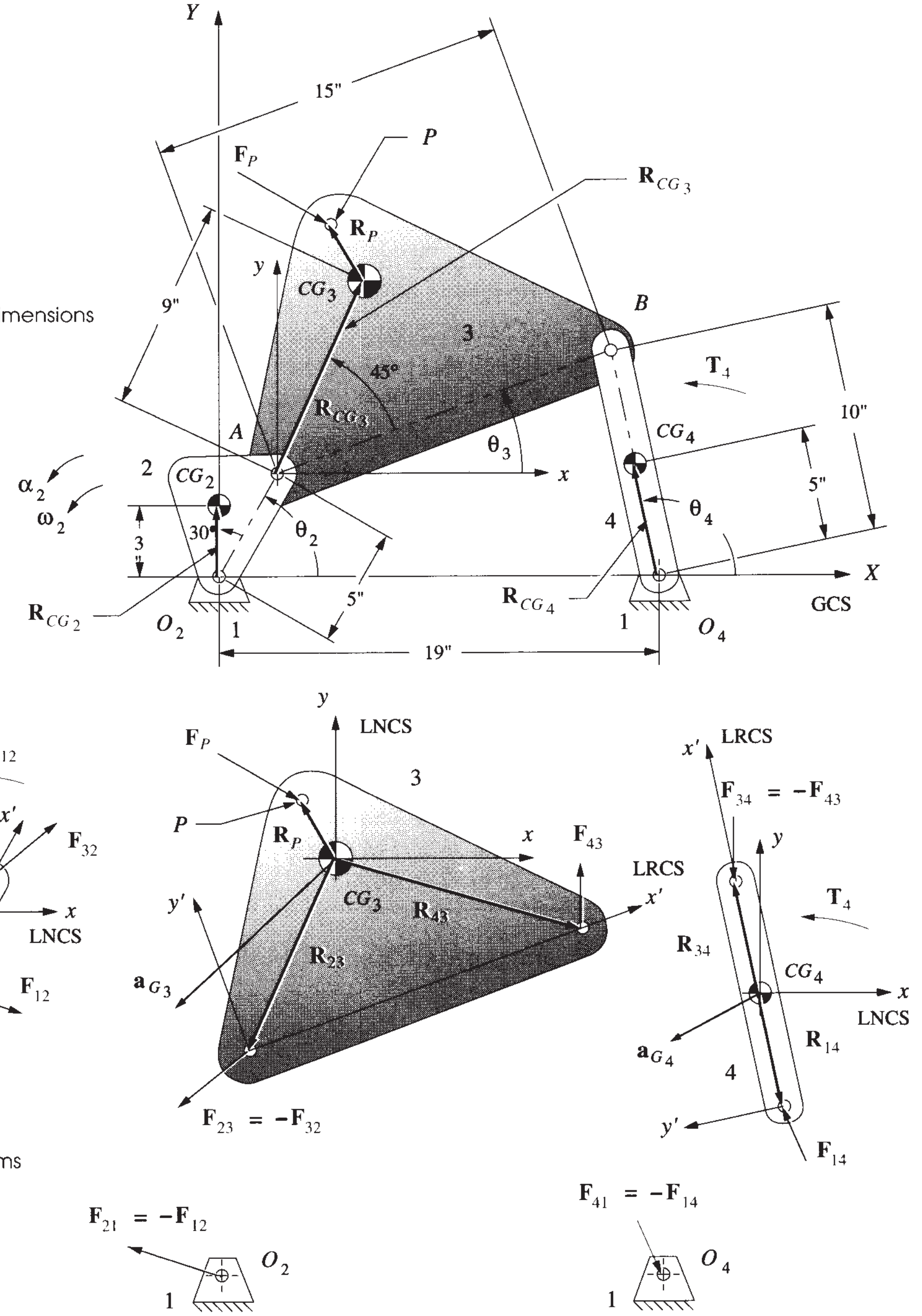
Read the disk file E011-02.mat into program MATRIX to exercise this example.

11.4 FORCE ANALYSIS OF A FOURBAR LINKAGE

Figure 11-3a shows a fourbar linkage. All dimensions of link lengths, link positions, locations of the links' *CG*s, linear accelerations of those *CG*s, and link angular accelerations and velocities have been previously determined from a kinematic analysis. We now wish to find the forces acting at all the pin joints of the linkage for one or more positions. The procedure is exactly the same as that used in the above two examples. This linkage has three moving links. Equation 11.1 provides three equations for any link or rigid body in motion. We should expect to have nine equations in nine unknowns for this problem.

Figure 11-3b shows the free-body diagrams for all links, with all forces shown. Note that an external force $\mathbf{F}_P$ is shown acting on link 3 at point *P*. Also an external torque $\mathbf{T}_4$ is shown acting on link 4. These external loads are due to some other mechanism (de-

(a) The linkage and dimensions



(b) Free-body diagrams

FIGURE 11-3

Dynamic force analysis of a fourbar linkage. (See also Figure P11-2, p. 561) vice, person, thing, etc.) pushing or twisting against the motion of the linkage. Any link can have any number of external loads and torques acting on it. Only one external torque and one external force are shown here to serve as examples of how they are handled in the computation. (Note that a more complicated force system, if present, could also be reduced to the combination of a single force and torque on each link.)

To solve for the pin forces it is necessary that these applied external forces and torques be defined for all positions of interest. We will solve for one member of the pair of action-reaction forces at each joint, and also for the driving torque $\mathbf{T}_{12}$ needed to be supplied at link 2 in order to maintain the kinematic state as defined. The force subscript convention is the same as that defined in the previous example. For example, $\mathbf{F}_{12}$ is the force of 1 *on* 2 and $\mathbf{F}_{32}$ is the force of 3 *on* 2. The equal and opposite forces at each of these pins are designated $\mathbf{F}_{21}$ and $\mathbf{F}_{23}$, respectively. All the unknown forces in the figure are shown at arbitrary angles and lengths as their true values are still to be determined.

The linkage kinematic parameters are defined with respect to a global *XY* system (GCS) whose origin is at the driver pivot O_2 and whose *X* axis goes through link 4's fixed pivot O_4. The mass (m) of each link, the location of its *CG*, and its mass moment of inertia (I_G) about that *CG* are also needed. The *CG* of each link is initially defined within each link with respect to a local moving and rotating axis system (LRCS) embedded in the link because the CG is an unchanging physical property of the link. The origin of this x',y' axis system is at one pin joint and the x' axis is the line of centers of the link. The *CG* position within the link is defined by a position vector in this LRCS. The instantaneous location of the *CG* can easily be determined for each dynamic link position by adding the angle of the internal *CG* position vector to the current GCS angle of the link.

We need to define each link's dynamic parameters and force locations with respect to a local, moving, but nonrotating axis system (LNCS) x,y located at its *CG* as shown on each free-body diagram in Figure 11-3b. The position vector locations of all attachment points of other links and points of application of external forces must be defined with respect to this LNCS axis system. These kinematic and applied force data differ for each position of the linkage. In the following discussion and examples, only one linkage position will be addressed. The process is identical for each succeeding position.

Equations 11.1 (p. 522) are now written for each moving link. For link 2, the result is identical to that done for the slider-crank example in equation 11.6a (p. 528).

$$\begin{aligned} F_{12_x} + F_{32_x} &= m_2 a_{G_{2_x}} \\ F_{12_y} + F_{32_y} &= m_2 a_{G_{2_y}} \\ T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{32_x} F_{32_y} - R_{32_y} F_{32_x}\right) &= I_{G_2} \alpha_2 \end{aligned} \tag{11.8a}$$

For link 3, with substitution of the reaction force $-\mathbf{F}_{32}$ for $\mathbf{F}_{23}$, the result is similar to equation 11.6b with some subscript changes to reflect the presence of link 4.

$$\begin{aligned} F_{43_x} - F_{32_x} + F_{P_x} &= m_3 a_{G_{3_x}} \\ F_{43_y} - F_{32_y} + F_{P_y} &= m_3 a_{G_{3_y}} \\ \left(R_{43_x} F_{43_y} - R_{43_y} F_{43_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) &= I_{G_3} \alpha_3 \end{aligned} \tag{11.8b}$$

For link 4, substituting the reaction force $-\mathbf{F}_{43}$ for $\mathbf{F}_{34}$, a similar set of equations 11.1 can be written:

$$\begin{aligned} F_{14_x} - F_{43_x} &= m_4 a_{G_{4x}} \\ F_{14_y} - F_{43_y} &= m_4 a_{G_{4y}} \\ \left(R_{14_x} F_{14_y} - R_{14_y} F_{14_x}\right) - \left(R_{34_x} F_{43_y} - R_{34_y} F_{43_x}\right) + T_4 &= I_{G_4} \alpha_4 \end{aligned} \tag{11.8c}$$

Note again that $\mathbf{T}_{12}$, the source torque, only appears in the equation for link 2 as that is the driver crank to which the motor is attached. Link 3, in this example, has no externally applied torque (though it could have) but does have an external force $\mathbf{F}_P$. Link 4, in this example, has no external force acting on it (though it could have) but does have an external torque $\mathbf{T}_4$. (The driving link 2 could also have an externally applied force on it though it usually does not.) There are nine unknowns present in these nine equations, F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14x}, F_{14y}, and T_{12}, so we can solve them simultaneously. We rearrange terms in equations 11.8 to put all known constant terms on the right side and then put them in matrix form.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & -R_{43_y} & R_{43_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R_{34_y} & -R_{34_x} & -R_{14_y} & R_{14_x} & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2} \alpha_2 \\ m_3 a_{G_{3x}} - F_{P_x} \\ m_3 a_{G_{3y}} - F_{P_y} \\ I_{G_3} \alpha_3 - R_{P_x} F_{P_y} + R_{P_y} F_{P_x} \\ m_4 a_{G_{4x}} \\ m_4 a_{G_{4y}} \\ I_{G_4} \alpha_4 - T_4 \end{bmatrix} \tag{11.9}$$

This system can be solved by using program MATRIX or any matrix solving calculator. As an example of this solution consider the following linkage data.

EXAMPLE 11-3

Dynamic Force Analysis of a Fourbar Linkage. (See Figure 11-3, p. 532)

Given: The 5-in-long crank (link 2) shown weighs 1.5 lb. Its *CG* is at 3 in at +30° from the line of centers. Its mass moment of inertia about its *CG* is 0.4 lb-in-sec^2. Its kinematic data are:

θ_2 deg	ω_2 rad/sec	α_2 rad/sec^2	a_{G_2} in/sec^2
60	25	–40	1878.84 @ 273.66°

The coupler (link 3) is 15 in long and weighs 7.7 lb. Its *CG* is at 9 in at 45° off the line of centers. Its mass moment of inertia about its *CG* is 1.5 lb-in-sec^2. Its kinematic data are:

θ_3 deg	ω_3 rad/sec	α_3 rad/sec^2	a_{G_3} in/sec^2
20.92	–5.87	120.9	3646.1 @ 226.5°

There is an external force of 80 lb at 330° on link 3, applied at point *P* which is located 3 in at 100° from the *CG* of link 3. There is an external torque on link 4 of 120 lb-in. The ground link is 19 in long. The rocker (link 4) is 10 in long and weighs 5.8 lb. Its *CG* is at 5 in at 0° off the line of centers. Its mass moment of inertia about its *CG* is 0.8 lb-in-sec^2. Its kinematic data are:

θ_4 deg	ω_4 rad/sec	α_4 rad/sec^2	a_{G_4} in/sec^2
104.41	7.93	276.29	1416.8 @ 207.2°

Find: The forces $\mathbf{F}_{12}$, $\mathbf{F}_{32}$, $\mathbf{F}_{43}$, $\mathbf{F}_{14}$, at the joints and the driving torque $\mathbf{T}_{12}$ needed to maintain motion with the given acceleration for this instantaneous position of the link.

Solution:

1 Convert the given weight to proper mass units, in this case blobs:

$$mass_{link2} = \frac{weight}{g} = \frac{1.5\text{ lb}}{386\text{ in/sec}^2} = 0.004\text{ blobs} \qquad (a)$$

$$mass_{link3} = \frac{weight}{g} = \frac{7.7\text{ lb}}{386\text{ in/sec}^2} = 0.020\text{ blobs} \qquad (b)$$

$$mass_{link4} = \frac{weight}{g} = \frac{5.8\text{ lb}}{386\text{ in/sec}^2} = 0.015\text{ blobs} \qquad (c)$$

2 Set up an LNCS *xy* coordinate system at the *CG* of each link, and draw all applicable vectors acting on that system as shown in the figure. Draw a free-body diagram of each moving link as shown.

3 Calculate the *x* and *y* components of the position vectors $\mathbf{R}_{12}$, $\mathbf{R}_{32}$, $\mathbf{R}_{23}$, $\mathbf{R}_{43}$, $\mathbf{R}_{34}$, $\mathbf{R}_{14}$, and $\mathbf{R}_P$ in the link's LNCS. $\mathbf{R}_{43}$, $\mathbf{R}_{34}$, and $\mathbf{R}_{14}$ will have to be calculated from the given link geometry data using the law of cosines and law of sines. Note that the current value of link 3's position angle (θ_3) in the GCS must be added to the angles of all position vectors before creating their *x,y* components in the LNCS if their angles were originally measured with respect to the link's embedded, local rotating coordinate system (LRCS).

$$
\begin{aligned}
\mathbf{R}_{12} &= 3.00 \ @ \angle\ 270.00°; & R_{12_x} &= 0.000, & R_{12_y} &= -3 \\
\mathbf{R}_{32} &= 2.83 \ @ \angle\ 28.00°; & R_{32_x} &= 2.500, & R_{32_y} &= 1.333 \\
\mathbf{R}_{23} &= 9.00 \ @ \angle\ 245.92°; & R_{23_x} &= -3.672, & R_{23_y} &= -8.217 \\
\mathbf{R}_{43} &= 10.72 \ @ \angle\ -15.46°; & R_{43_x} &= 10.332, & R_{43_y} &= -2.858 \\
\mathbf{R}_{34} &= 5.00 \ @ \angle\ 104.41°; & R_{34_x} &= -1.244, & R_{34_y} &= 4.843 \\
\mathbf{R}_{14} &= 5.00 \ @ \angle\ 284.41°; & R_{14_x} &= 1.244, & R_{14_y} &= -4.843 \\
\mathbf{R}_{P} &= 3.00 \ @ \angle\ 120.92°; & R_{P_x} &= -1.542, & R_{P_y} &= 2.574
\end{aligned} \qquad (d)
$$

4 Calculate the x and y components of the acceleration of the CGs of all moving links in the global coordinate system (GCS):

$$
\begin{aligned}
\mathbf{a}_{G_2} &= 1878.84 \ @ \ \angle 273.66°; & a_{G_{2x}} &= 119.94, & a_{G_{2y}} &= -1875.01 \\
\mathbf{a}_{G_3} &= 3646.10 \ @ \ \angle 226.51°; & a_{G_{3x}} &= -2509.35, & a_{G_{3y}} &= -2645.23 \\
\mathbf{a}_{G_4} &= 1416.80 \ @ \ \angle 207.24°; & a_{G_{4x}} &= -1259.67, & a_{G_{4y}} &= -648.50
\end{aligned} \qquad (e)
$$

5 Calculate the x and y components of the external force at P in the GCS:

$$
\mathbf{F}_{P3} = 80 \ @ \ \angle\ 330°; \qquad F_{P3_x} = 69.28, \qquad F_{P3_y} = -40.00 \qquad (f)
$$

6 Substitute these given and calculated values into the matrix equation 11.9 (p. 534).

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & -1.330 & 2.5 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -8.217 & 3.673 & 2.861 & 10.339 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4.843 & 1.244 & 4.843 & 1.244 & 0
\end{bmatrix}
\times
\begin{bmatrix}
F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12}
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
(0.004)(119.94) \\
(0.004)(-1875.01) \\
(0.4)(-40) \\
(0.02)(-2509.35)-(69.28) \\
(0.02)(-2645.23)-(-40) \\
(1.5)(120.9)-\left[(-1.542)(-40)-(2.574)(69.28)\right] \\
(0.015)(-1259.67) \\
(0.015)(-648.50) \\
(0.8)(276.29)-(120)
\end{bmatrix}
=
\begin{bmatrix}
0.480 \\ -7.500 \\ -16.000 \\ -119.465 \\ -12.908 \\ 298.003 \\ -18.896 \\ -9.727 \\ 101.031
\end{bmatrix} \qquad (g)
$$

7 Solve this system either by inverting matrix A and premultiplying that inverse times matrix C using a pocket calculator such as the HP-28, or by inputting the values for matrices A and C to program MATRIX provided with this text which gives the following solution:

$$\begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -117.65 \\ -107.84 \\ 118.13 \\ 100.34 \\ -1.34 \\ 87.43 \\ -20.23 \\ 77.71 \\ 243.23 \end{bmatrix} \qquad (h)$$

Converting the forces to polar coordinates:

$$\begin{aligned} \mathbf{F}_{12} &= 159.60 \text{ lb} \ @ \angle \ 222.52° \\ \mathbf{F}_{32} &= 154.99 \text{ lb} \ @ \angle \ 40.35° \\ \mathbf{F}_{43} &= 87.44 \text{ lb} \ @ \angle \ 90.88° \\ \mathbf{F}_{14} &= 80.30 \text{ lb} \ @ \angle \ 104.59° \end{aligned} \qquad (i)$$

8 The pin-force magnitudes in (i) are needed to size the pivot pins and links against failure and to select pivot bearings that will last for the required life of the assembly. The driving torque T_{12} defined in *(h)* is needed to select a motor or other device capable of supplying the power to drive the system. See Section 2.16 (p. 60) for a brief discussion of motor selection. Issues of stress calculation and failure prevention are beyond the scope of this text, but note that those calculations cannot be done until a good estimate of the dynamic forces and torques on the system has been made by methods such as those shown in this example.

This solves the linkage for one position. A new set of values can be put into the A and C matrices for each position of interest at which a force analysis is needed. Read the disk file EII-03.mat into program MATRIX to exercise this example. The disk file EII-03.4br can also be read into program FOURBAR which will run the linkage through a series of positions starting with the stated parameters as initial conditions. The linkage will slow to a stop and then run in reverse due to the negative acceleration. The matrix of equation *(g)* can be seen within FOURBAR using *Dynamics/Solve/Show Matrix.*

It is worth noting some general observations about this method at this point. The solution is done using cartesian coordinates of all forces and position vectors. Before being placed in the matrices, these vector components must be defined in the global coordinate system (GCS) or in nonrotating, local coordinate systems, parallel to the global coordinate system, with their origins at the links' *CGs* (LNCS). Some of the linkage parameters are normally expressed in such coordinate systems, but some are not, and so must be converted. The kinematic data should all be computed in the global system or in parallel, **nonrotating,** local systems placed at the *CGs* of individual links. Anyexternal forces on the links must also be defined in the global system.

However, the position vectors that define intralink locations, such as the pin joints versus the *CG*, or which locate points of application of external forces versus the *CG* are defined in local, **rotating** coordinate systems embedded in the links (LRCS). Thus these position vectors must be redefined in a **nonrotating**, parallel system before being used in the matrix. An example of this is vector $\mathbf{R}_p$, which was initially defined as 3 in at 100° in link 3's embedded, **rotating** coordinate system. Note in the example above that its cartesian coordinates for use in the equations were calculated after adding the current value of θ_3 to its angle. This redefined $\mathbf{R}_p$ as 3 in at 120.92° in the **nonrotating** local system. The same was done for position vectors $\mathbf{R}_{12}$, $\mathbf{R}_{32}$, $\mathbf{R}_{23}$, $\mathbf{R}_{43}$, $\mathbf{R}_{34}$, and $\mathbf{R}_{14}$. In each case the **intralink angle** of these vectors (which is independent of linkage position) was added to the current link angle to obtain its position in the *xy* system at the link's *CG*. The proper definition of these position vector components is critical to the solution, and it is very easy to make errors in defining them.

To further confuse things, even though the position vector $\mathbf{R}_p$ is initially measured in the link's embedded, rotating coordinate system, the force $\mathbf{F}_p$, which it locates, is not. The force $\mathbf{F}_p$ is not part of the link, as is $\mathbf{R}_p$, but rather is part of the external world, so it is defined in the global system.

11.5 FORCE ANALYSIS OF A FOURBAR SLIDER-CRANK LINKAGE

The approach taken for the pin-jointed fourbar is equally valid for a fourbar slider-crank linkage. The principal difference will be that the slider block will have no angular acceleration. Figure 11-4 shows a fourbar slider-crank with an external force on the slider block, link 4. This is representative of the mechanism used extensively in piston pumps and internal combustion engines. We wish to determine the forces at the joints and the driving torque needed on the crank to provide the specified accelerations. A kinematic analysis must have previously been done in order to determine all position, velocity, and acceleration information for the positions being analyzed. Equations 11.1 are written for each link. For link 2:

$$\begin{aligned} F_{12_x} + F_{32_x} &= m_2 a_{G_{2x}} \\ F_{12_y} + F_{32_y} &= m_2 a_{G_{2y}} \\ T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{32_x} F_{32_y} - R_{32_y} F_{32_x}\right) &= I_{G_2} \alpha_2 \end{aligned} \tag{11.10a}$$

This is identical to equation 11.8a for the "pure" fourbar linkage. For link 3:

$$\begin{aligned} F_{43_x} - F_{32_x} &= m_3 a_{G_{3x}} \\ F_{43_y} - F_{32_y} &= m_3 a_{G_{3y}} \\ \left(R_{43_x} F_{43_y} - R_{43_y} F_{43_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) &= I_{G_3} \alpha_3 \end{aligned} \tag{11.10b}$$

This is similar to equation 11.8b, lacking only the terms involving $\mathbf{F}_p$ since there is no external force shown acting on link 3 of our example slider-crank. For link 4:

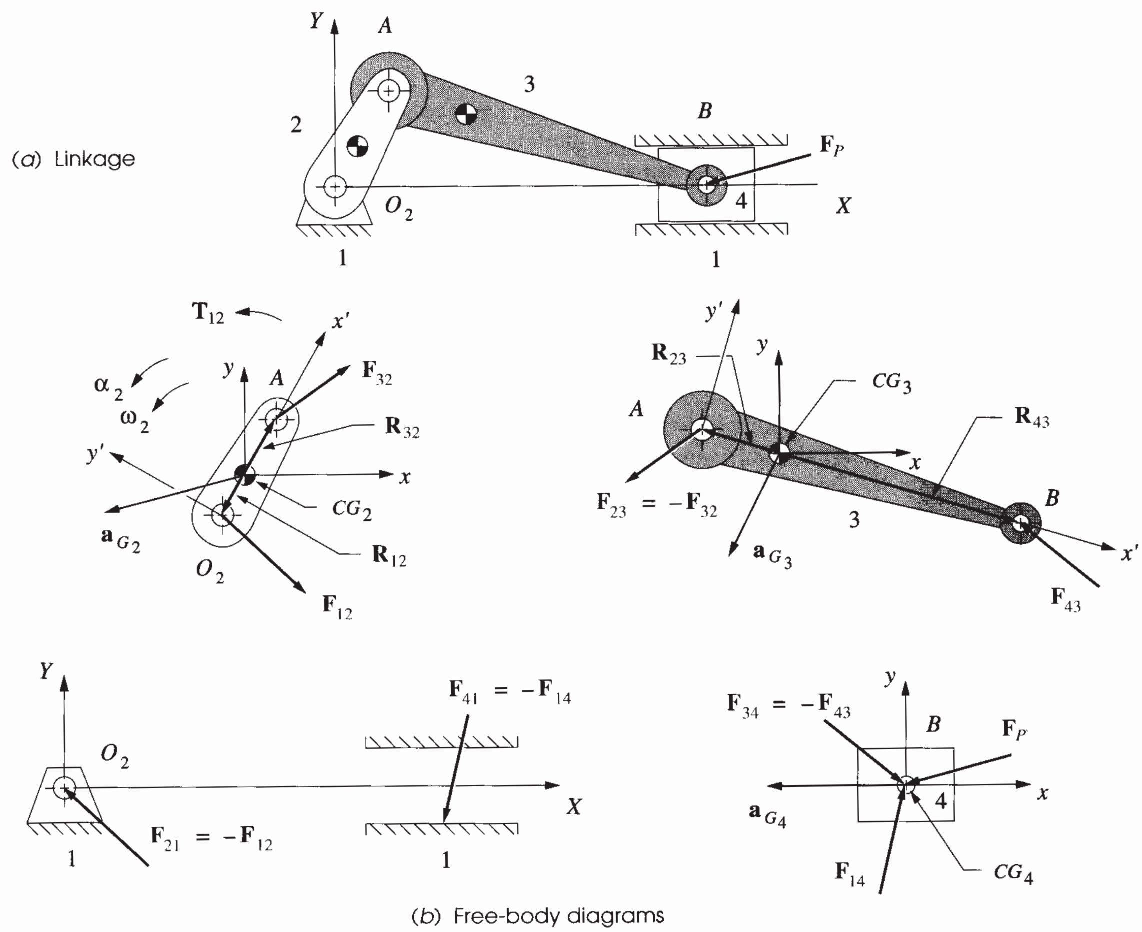


FIGURE 11-4

Dynamic force analysis of the fourbar slider-crank linkage

$$F_{14_x} - F_{43_x} + F_{P_x} = m_4 a_{G4_x}$$

$$F_{14_y} - F_{43_y} + F_{P_y} = m_4 a_{G4_y} \tag{11.10c}$$

$$\left(R_{14_x} F_{14_y} - R_{14_y} F_{14_x}\right) - \left(R_{34_x} F_{43_y} - R_{34_y} F_{43_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) = I_{G_4} \alpha_4$$

These contain the external force $\mathbf{F}_p$ shown acting on link 4.

For the inversion of the slider-crank shown, the slider block, or piston, is in pure translation against the stationary ground plane; thus it can have no angular acceleration or angular velocity. Also, the position vectors in the torque equation (equation 11.10c) are all zero as the force $\mathbf{F}_p$ acts at the *CG*. Thus the torque equation for link 4 (third expression in equation 11.10c) is zero for this inversion of the slider-crank linkage. Its linear acceleration also has no *y* component.

$$\alpha_4 = 0, \qquad a_{G_{4_y}} = 0 \tag{11.10d}$$

The only x directed force that can exist at the interface between links 4 and 1 is friction. Assuming coulomb friction, the x component can be expressed in terms of the y component of force at this interface. We can write a relation for the friction force f at that interface such as $f = \pm\mu\mathbf{N}$, where $\pm\mu$ is a known coefficient of friction. The plus and minus signs on the coefficient of friction are to recognize the fact that the friction force always opposes motion. The kinematic analysis will provide the velocity of the link at the sliding joint. The sign of μ will always be the opposite of the sign of this velocity.

$$F_{14_x} = \pm\mu F_{14_y} \tag{11.10e}$$

Substituting equations 11.10d and 11.10e into the reduced equation 11.10c yields:

$$\begin{aligned} \pm\mu F_{14_y} - F_{43_x} + F_{P_x} &= m_4 a_{G_{4_x}} \\ F_{14_y} - F_{43_y} + F_{P_y} &= 0 \end{aligned} \tag{11.10f}$$

This last substitution has reduced the unknowns to eight, F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14y}, and T_{12}; thus we need only eight equations. We can now use the eight equations in 11.10a, b, and f to assemble the matrices for solution.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & -R_{43_y} & R_{43_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \pm\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2_x}} \\ m_2 a_{G_{2_y}} \\ I_{G_2}\alpha_2 \\ m_3 a_{G_{3_x}} \\ m_3 a_{G_{3_y}} \\ I_{G_3}\alpha_3 \\ m_4 a_{G_{4_x}} - F_{P_x} \\ -F_{P_y} \end{bmatrix} \tag{11.10g}$$

Solution of this matrix equation 11.10g plus equation 11.10e will yield complete dynamic force information for the fourbar slider-crank linkage.

11.6 FORCE ANALYSIS OF THE INVERTED SLIDER-CRANK

Another inversion of the fourbar slider-crank was also analyzed kinematically in Part I. It is shown in Figure 11-5. Link 4 does have an angular acceleration in this inversion. In fact, it must have the same angle, angular velocity, and angular acceleration as link 3 because they are rotationally coupled by the sliding joint. We wish to determine the forces at all pin joints and at the sliding joint as well as the driving torque needed to create the desired accelerations. Each link's joints are located by position vectors referenced to nonrotating local *xy* coordinate systems at each link's *CG* as before. The sliding joint is located by the position vector $\mathbf{R}_{43}$ to the center of the slider, point *B*. The instantaneous position of point *B* was determined from the kinematic analysis as length *b* referenced to instant center I_{23} (point *A*). See Sections 4.7 (p. 161), 6.7 (p. 276), and 7.3 (p. 315) to review the position, velocity, and acceleration analysis of this mechanism. Recall that this mechanism has a nonzero Coriolis component of acceleration. The force between link 3 and link 4 within the sliding joint is distributed along the unspecified length of the slider block. For this analysis the distributed force can be modeled as a force concentrated at point *B* within the sliding joint. We will neglect friction in this example.

The equations for links 2 and 3 are identical to those for the noninverted slider-crank (Equations 11.10a and b). The equations for link 4 are the same as equations 11.10c except for the absence of the terms involving $\mathbf{F}_p$ since no external force is shown acting on link 4 in this example. The slider joint can only transmit force from link 3 to link 4 or vice versa along a line perpendicular to the axis of slip. This line is called the axis of transmission. In order to guarantee that the force $\mathbf{F}_{34}$ or $\mathbf{F}_{43}$ is always perpendicular to the axis of slip, we can write the following relation:

$$\hat{\mathbf{u}} \cdot \mathbf{F}_{43} = 0 \tag{11.11a}$$

which expands to:

$$u_x F_{43_x} + u_y F_{43_y} = 0 \tag{11.11b}$$

The dot product of two vectors will be zero when the vectors are mutually perpendicular. The unit vector *u hat* is in the direction of link 3 which is defined from the kinematic analysis as θ_3.

$$u_x = \cos\theta_3, \qquad u_y = \sin\theta_3 \tag{11.11c}$$

Equation 11.11 provides a tenth equation, but we have only nine unknowns, F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14x}, F_{14y}, and T_{12}, so one of our equations is redundant. Since we must include equation 11.11, we will combine the torque equations for links 3 and 4 rewritten here in vector form and without the external force $\mathbf{F}_p$.

$$\begin{gathered}(\mathbf{R}_{43} \times \mathbf{F}_{43}) - (\mathbf{R}_{23} \times \mathbf{F}_{32}) = I_{G_3}\alpha_3 = I_{G_3}\alpha_4 \\ (\mathbf{R}_{14} \times \mathbf{F}_{14}) - (\mathbf{R}_{34} \times \mathbf{F}_{43}) = I_{G_4}\alpha_4\end{gathered} \tag{11.12a}$$

Note that the angular acceleration of link 3 is the same as that of link 4 in this linkage. Adding these equations gives:

$$(\mathbf{R}_{43} \times \mathbf{F}_{43}) - (\mathbf{R}_{23} \times \mathbf{F}_{32}) + (\mathbf{R}_{14} \times \mathbf{F}_{14}) - (\mathbf{R}_{34} \times \mathbf{F}_{43}) = (I_{G_3} + I_{G_4})\alpha_4 \tag{11.12b}$$

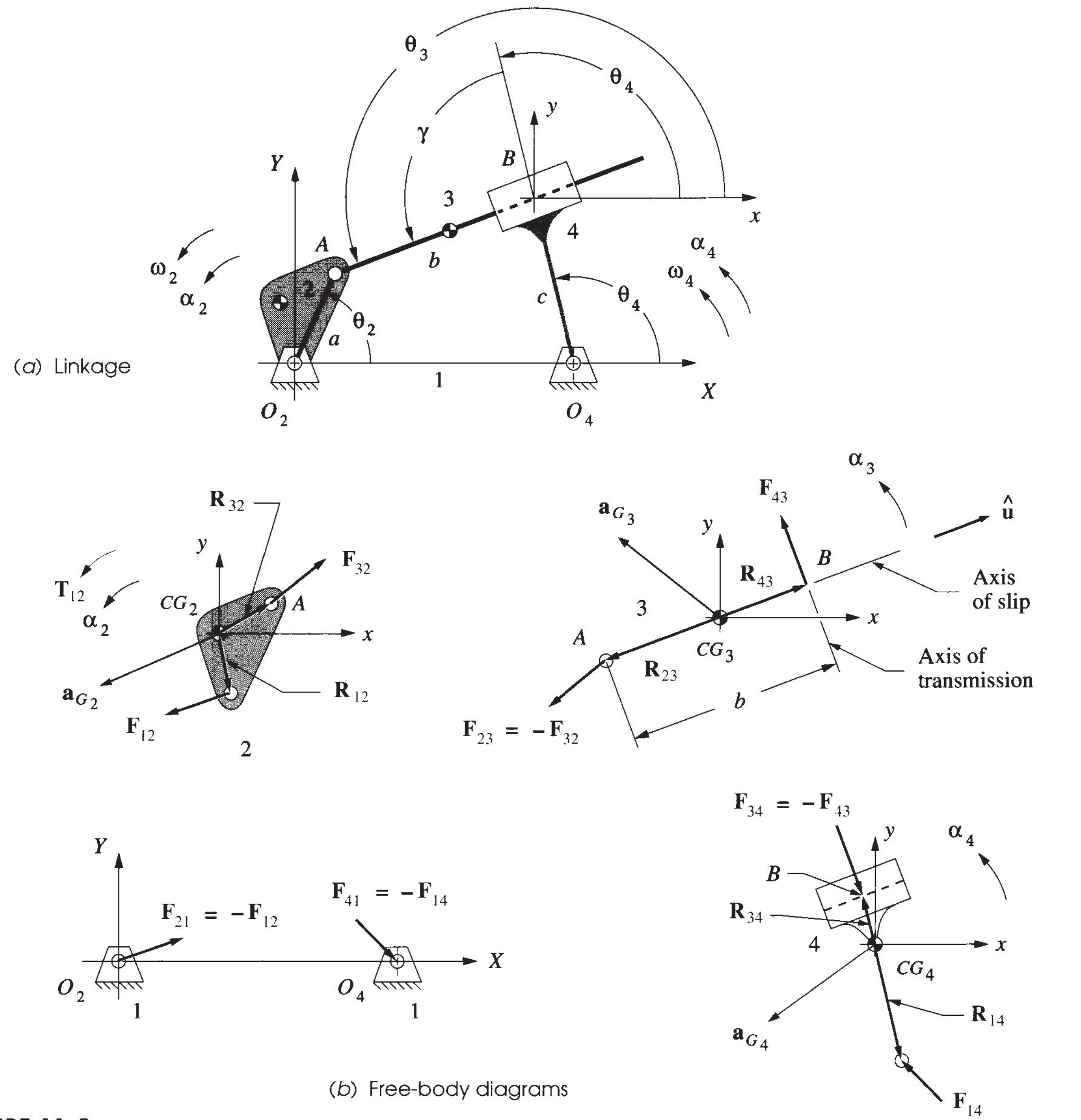


FIGURE 11-5

Dynamic forces in the inverted slider-crank fourbar linkage

Expanding and collecting terms:

$$\begin{aligned}&\left(R_{43_x}-R_{34_x}\right)F_{43_y}+\left(R_{34_y}-R_{43_y}\right)F_{43_x}-R_{23_x}F_{32_y}\\&+R_{23_y}F_{32_x}+R_{14_x}F_{14_y}-R_{14_y}F_{14_x}=\left(I_{G_3}+I_{G_4}\right)\alpha_4\end{aligned} \tag{11.12c}$$

Equations 11.10a, 11.11b, 11.12c, and the four force equations from equations 11.10b and 11.10c (excluding the external force $\mathbf{F}_P$) give us nine equations in the nine unknowns which we can put in matrix form for solution.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & \left(R_{34_y} - R_{43_y}\right) & \left(R_{43_x} - R_{34_x}\right) & -R_{14_y} & R_{14_x} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & u_x & u_y & 0 & 0 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2} \alpha_2 \\ m_3 a_{G_{3x}} \\ m_3 a_{G_{3y}} \\ \left(I_{G_3} + I_{G_4}\right)\alpha_4 \\ m_4 a_{G_{4x}} \\ m_4 a_{G_{4y}} \\ 0 \end{bmatrix} \tag{11.13}$$

11.7 FORCE ANALYSIS—LINKAGES WITH MORE THAN FOUR BARS

This matrix method of force analysis can easily be extended to more complex assemblages of links. The equations for each link are of the same form. We can create a more general notation for equations 11.1 to apply them to any assembly of n pin-connected links. Let j represent any link in the assembly. Let $i = j - 1$ be the previous link in the chain and $k = j + 1$ be the next link in the chain; then, using the vector form of equations 11.1:

$$\mathbf{F}_{ij} + \mathbf{F}_{jk} + \sum \mathbf{F}_{ext_j} = m_j \, \mathbf{a}_{G_j} \tag{11.14a}$$

$$\left(\mathbf{R}_{ij} \times \mathbf{F}_{ij}\right) + \left(\mathbf{R}_{jk} \times \mathbf{F}_{jk}\right) + \sum \mathbf{T}_j + \left(\mathbf{R}_{ext_j} \times \sum \mathbf{F}_{ext_j}\right) = I_{G_j} \alpha_j \tag{11.14b}$$

where:

$$j = 2, 3, \ldots, n; \qquad i = j - 1; \qquad k = j + 1, \; j \neq n; \qquad \text{if } j = n, \; k = 1$$

and

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}; \qquad \mathbf{F}_{kj} = -\mathbf{F}_{jk} \tag{11.14c}$$

The sum of forces vector equation 11.14a can be broken into its two x and y component equations and then applied, along with the sum of torques equation 11.14b, to each of the links in the chain to create the set of simultaneous equations for solution. Any link may have external forces and/or external torques applied to it. All will have pin forces. Since the *n*th link in a closed chain connects to the first link, the value of *k* for the *n*th link is set to 1. In order to reduce the number of variables to a tractable quantity, substitute the negative reaction forces from equation 11.14c where necessary as was done in the examples above. When sliding joints are present, it will be necessary to add constraints on the allowable directions of forces at those joints as was done in the inverted slider-crank example above.

11.8 SHAKING FORCES AND SHAKING TORQUE

It is usually of interest to know the net effect of the dynamic forces as felt on the ground plane as this can set up vibrations in the structure which supports the machine. For our simple examples of three- and fourbar linkages, there are only two points at which the dynamic forces can be delivered to link 1, the ground plane. More complicated mechanisms will have more joints with the ground plane. The forces delivered by the moving links to the ground at the fixed pivots O_2 and O_4 are designated $\mathbf{F}_{21}$ and $\mathbf{F}_{41}$ by our subscript convention as defined in Section 11.1 (p. 521). Since we chose to solve for $\mathbf{F}_{12}$ and $\mathbf{F}_{14}$ in our solutions, we simply negate those forces to obtain their equal and opposite counterparts (see also equation 11.5, p. 527).

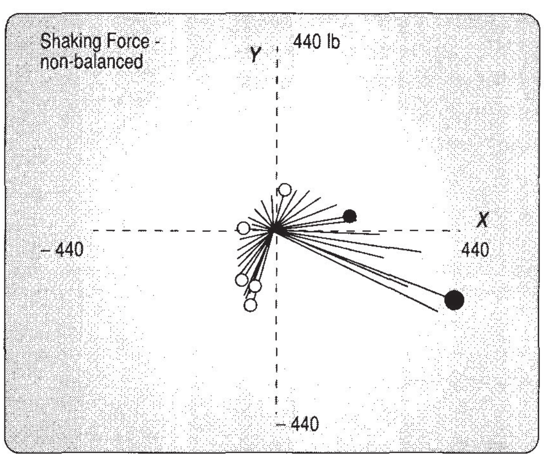
$$\mathbf{F}_{21} = -\mathbf{F}_{12} \qquad \mathbf{F}_{41} = -\mathbf{F}_{14} \tag{11.15a}$$

The *sum of all the forces acting on the ground plane* is called the **shaking force** ($\mathbf{F}_s$) as shown in Figure 11-6.* In these simple examples it is equal to:

$$\mathbf{F}_s = \mathbf{F}_{21} + \mathbf{F}_{41} \tag{11.15b}$$

The *reaction torque felt by the ground plane* is also called the **shaking torque** ($\mathbf{T}_s$) as shown in Figure 11-7.* This is the negative of the source torque $\mathbf{T}_{12}$ which is delivered to the driving link from the ground.

* The FOURBAR disk file (F11-06.4br) that generated the plots in Figures 11-6 and 11-7 may be opened in that program to see more detail on the linkage's dynamics.



Link No.	Length in	Mass Units	Inertia Units	CG Posit	at Deg	Ext. Force lb	at Deg
1	5.5						
2	2.0	.002	.004	1.0	0		
3	6.0	.030	.060	2.5	30	12	270
4	3.0	.010	.020	1.5	0	60	–45

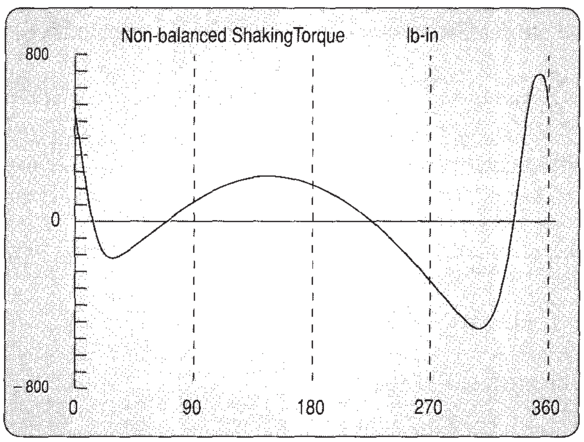
Coupler pt. = 3 in @ 45°
Open/Crossed = open

Ext. Force 3 acts at 5 in @ 30° vs. CG of Link 3
Ext. Force 4 acts at 5 in @ 90° vs. CG of Link 4
Ext. Torque 3 = – 20 lb-in
Ext. Torque 4 = 25 lb-in

Start Alpha2 = 0 rad/sec^2
Start Omega2 = 50 rad/sec
Start Theta2 = 0°
Final Theta2 = 360°
Delta Theta2 = 10°

FIGURE 11-6

Linkage data and polar plot of shaking force for an unbalanced crank-rocker fourbar linkage from program FOURBAR



Link No.	Length in	Mass Units	Inertia Units	CG Posit	at Deg	Ext. Force lb	at Deg
1	5.5						
2	2.0	.002	.004	1.0	0		
3	6.0	.030	.060	2.5	30	12	270
4	3.0	.010	.020	1.5	0	60	–45

Coupler pt. = 3 in @ 45°
Open/Crossed = open

Ext. Force 3 acts at 5 in @ 30° vs. CG of Link 3
Ext. Force 4 acts at 5 in @ 90° vs. CG of Link 4
Ext. Torque 3 = – 20 lb-in
Ext. Torque 4 = 25 lb-in

Start Alpha2 = 0 rad/sec^2
Start Omega2 = 50 rad/sec
Start Theta2 = 0°
Final Theta2 = 360°
Delta Theta2 = 2°

FIGURE 11-7

Linkage data and shaking torque curve for an unbalanced crank-rocker fourbar linkage from program FOURBAR

$$\mathbf{T}_s = \mathbf{T}_{21} = -\mathbf{T}_{12} \tag{11.15c}$$

The shaking force will tend to move the ground plane back and forth, and the shaking torque will tend to rock the ground plane about the driveline axis. Both will cause vibrations. We are usually looking to minimize the effects of the shaking forces and shaking torques on the frame. This can sometimes be done by balancing, sometimes by the addition of a flywheel to the system, and sometimes by shock mounting the frame to isolate the vibrations from the rest of the assembly. Most often we will use a combination of all three approaches. We will investigate some of these techniques shortly.

11.9 PROGRAM FOURBAR

The matrix methods introduced in the preceding sections all provide force and torque information for one position of the linkage assembly as defined by its kinematic and geometric parameters. To do a complete force analysis for multiple positions of a machine requires that these computations be repeated with new input data for each position. A computer program is the obvious way to accomplish this. Program FOURBAR, on the enclosed CD-ROM, computes the kinematic parameters for any fourbar linkage over changes in time or driver (crank) angle plus the forces and torques concomitant with the linkage kinematics and link geometry. Examples of its output are shown in Figures 11-6 and 11-7. Please refer to Appendix A for information on the use of program FOURBAR.

11.10 LINKAGE FORCE ANALYSIS BY ENERGY METHODS

In Section 10.13 (p. 515) the method of virtual work was presented. We will now use that approach to solve the linkage from Example 11-3 as a check on its solution by the newtonian method used above. The kinematic data given in Example 11-3 did not include information on the angular velocities of all the links, the linear velocities of the centers of gravities of the links, and the linear velocity of the point P of application of the external force on link 3. Velocity data were not needed for the newtonian solution but are for the virtual work approach and are detailed below. Equation 10.26a is repeated here.

$$\sum_{k=2}^{n} \mathbf{F}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} \mathbf{T}_k \cdot \boldsymbol{\omega}_k = \sum_{k=2}^{n} m_k \mathbf{a}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} I_k \boldsymbol{\alpha}_k \cdot \boldsymbol{\omega}_k \tag{11.16a}$$

Expanding the summations, still in vector form:

$$\begin{aligned}\left(\mathbf{F}_{P_3} \cdot \mathbf{v}_{P_3} + \mathbf{F}_{P_4} \cdot \mathbf{v}_{P_4}\right) + \left(\mathbf{T}_{12} \cdot \boldsymbol{\omega}_2 + \mathbf{T}_3 \cdot \boldsymbol{\omega}_3 + \mathbf{T}_4 \cdot \boldsymbol{\omega}_4\right) &= \\ \left(m_2 \mathbf{a}_{G_2} \cdot \mathbf{v}_{G_2} + m_3 \mathbf{a}_{G_3} \cdot \mathbf{v}_{G_3} + m_4 \mathbf{a}_{G_4} \cdot \mathbf{v}_{G_4}\right) & \\ + \left(I_{G_2} \boldsymbol{\alpha}_2 \cdot \boldsymbol{\omega}_2 + I_{G_3} \boldsymbol{\alpha}_3 \cdot \boldsymbol{\omega}_3 + I_{G_4} \boldsymbol{\alpha}_4 \cdot \boldsymbol{\omega}_4\right) &\end{aligned} \tag{11.16b}$$

Expanding the dot products to create a scalar equation:

$$\begin{aligned}\left(F_{P_{3_x}} V_{P_{3_x}} + F_{P_{3_y}} V_{P_{3_y}}\right) + \left(F_{P_{4_x}} V_{P_{4_x}} + F_{P_{4_y}} V_{P_{4_y}}\right) + \left(T_{12}\omega_2 + T_3\omega_3 + T_4\omega_4\right) &= \\ m_2\left(a_{G_{2_x}} V_{G_{2_x}} + a_{G_{2_y}} V_{G_{2_y}}\right) + m_3\left(a_{G_{3_x}} V_{G_{3_x}} + a_{G_{3_y}} V_{G_{3_y}}\right) & \\ + m_4\left(a_{G_{4_x}} V_{G_{4_x}} + a_{G_{4_y}} V_{G_{4_y}}\right) + \left(I_{G_2}\alpha_2\omega_2 + I_{G_3}\alpha_3\omega_3 + I_{G_4}\alpha_4\omega_4\right) &\end{aligned} \tag{11.16c}$$

EXAMPLE 11-4

Analysis of a Fourbar Linkage by the Method of Virtual Work. (See Figure 11-3, p. 532.)

Given: The 5-in-long crank (link 2) shown weighs 1.5 lb. Its *CG* is at 3 in at +30° from the line of centers. Its mass moment of inertia about its *CG* is 0.4 lb-in-sec^2. Its kinematic data are:

θ_2 deg	ω_2 rad/sec	α_2 rad/sec^2	V_{G_2} in/sec
60	25	–40	75 @ 180°

The coupler (link 3) is 15 in long and weighs 7.7 lb. Its *CG* is at 9 in at 45° off the line of centers. Its mass moment of inertia about its *CG* is 1.5 lb-in-sec^2. Its kinematic data are:

θ_3 deg	ω_3 rad/sec	α_3 rad/sec^2	V_{G_3} in/sec
20.92	–5.87	120.9	72.66 @ 145.7°

There is an external force on link 3 of 80 lb at 330°, applied at point *P* which is located 3 in @ 100° from the *CG* of link 3. The linear velocity of that point is 67.2 in/sec at 131.94°.

The rocker (link 4) is 10 in long and weighs 5.8 lb. Its *CG* is at 5 in at 0° off the line of centers. Its mass moment of inertia about its *CG* is 0.8 lb-in-sec^2. Its kinematic data are:

θ_4 deg	ω_4 rad/sec	α_4 rad/sec^2	V_{G_4} in/sec
104.41	7.93	276.29	39.66 @ 194.41°

There is an external torque on link 4 of 120 lb-in. The ground link is 19 in long.

Find: The driving torque T_{12} needed to maintain motion with the given acceleration for this instantaneous position of the link.

Solution:

1 The torque, angular velocity, and angular acceleration vectors in this two-dimensional problem are all directed along the Z axis, so their dot products each have only one term. Note that in this particular example there is no force $\mathbf{F}_{P4}$ and no torque $\mathbf{T}_3$.

2 The cartesian coordinates of the acceleration data were calculated in Example 11-3 above.

$$\begin{aligned}
\mathbf{a}_{G_2} &= 1878.84 \ @\angle\ 273.66°; & a_{G_{2x}} &= 119.94, & a_{G_{2y}} &= -1875.01 \\
\mathbf{a}_{G_3} &= 3646.10 \ @\angle\ 226.51°; & a_{G_{3x}} &= -2509.35, & a_{G_{3y}} &= -2645.23 \\
\mathbf{a}_{G_4} &= 1416.80 \ @\angle\ 207.24°; & a_{G_{4x}} &= -1259.67, & a_{G_{4y}} &= -648.50
\end{aligned} \tag{a}$$

3 The x and y components of the external force at P in the global coordinate system were also calculated in Example 11-3:

$$\mathbf{F}_{P_3} = 80 \ @\ \angle 330°; \qquad F_{P_{3x}} = 69.28, \qquad F_{P_{3y}} = -40.00 \tag{b}$$

4 Converting the velocity data for this example to cartesian coordinates:

$$\begin{aligned}
\mathbf{V}_{G_2} &= 75 \ @\angle\ 180°; & V_{G_{2x}} &= -75, & V_{G_{2y}} &= 0 \\
\mathbf{V}_{G_3} &= 72.66 \ @\angle\ 145.70°; & V_{G_{3x}} &= -60.02, & V_{G_{3y}} &= 40.95 \\
\mathbf{V}_{G_4} &= 39.66 \ @\angle\ 194.41°; & V_{G_{4x}} &= -38.41, & V_{G_{4y}} &= -9.87 \\
\mathbf{V}_{P_3} &= 67.20 \ @\angle\ 131.94°; & V_{P_{3x}} &= -44.91, & V_{P_{3y}} &= 49.99
\end{aligned} \tag{c}$$

5 Substituting the example data into equation 11.16c:

$$\begin{aligned}
&\left[(69.28)(-44.91)+(-40)(49.99)\right]+[0]+\left[25T_{12}+(0)+(120)(7.93)\right]= \\
&\quad \frac{1.5}{386}\left[(119.94)(-75)+(-1875.01)(0)\right] \\
&\quad +\frac{7.7}{386}\left[(-2509.35)(-60.02)+(-2645.23)(40.95)\right] \\
&\quad +\frac{5.8}{386}\left[(-1259.67)(-38.41)+(-648.50)(-9.87)\right] \\
&\quad +\left[(0.4)(-40)(25)+(1.5)(120.9)(-5.87)+(0.8)(276.29)(7.93)\right]
\end{aligned} \tag{d}$$

6 The only unknown in this equation is the input torque $\mathbf{T}_{12}$ which calculates to:

$$\mathbf{T}_{12} = 243.2\,\hat{\mathbf{k}} \tag{e}$$

which is the same as the answer obtained in Example 11-3.

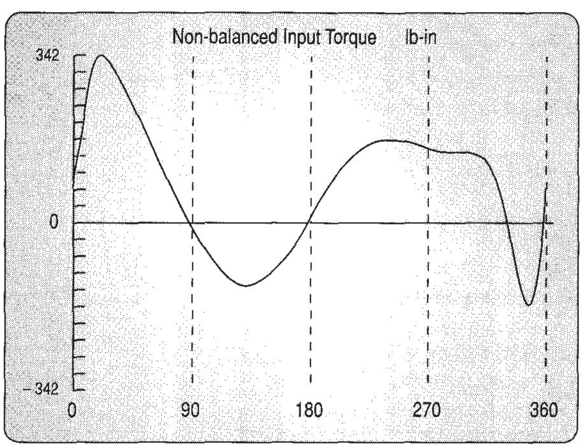
This method of virtual work is useful if a quick answer is needed for the input torque, but it does not give any information about the joint forces.

11.11 CONTROLLING INPUT TORQUE-flYWHEEIS

The typically large variation in accelerations within a mechanism can cause significant oscillations in the torque required to drive it at a constant or near constant speed. The peak torques needed may be so high as to require an overly large motor to deliver them. However, the average torque over the cycle, due mainly to losses and external work done, may often be much smaller than the peak torque. We would like to provide some means to smooth out these oscillations in torque during the cycle. This will allow us to size the motor to deliver the average torque rather than the peak torque. One convenient and relatively inexpensive means to this end is the addition of a flywheel to the system.

TORQUE VARIATION Figure 11-8 shows the variation in the input torque for a crank-rocker fourbar linkage over one full revolution of the drive crank. It is running at a constant angular velocity of 50 rad/sec. The torque varies a great deal within one cycle of the mechanism, going from a positive peak of 341.7 Ib-in to a negative peak of -166.41b-in. The average value of this torque over the cycle is only 70.21b-in, being due to the *external work done plus losses.* This linkage has only a 12-lb external force applied to link 3 at the CG and a 25 Ib-in external torque applied to link 4. These small external loads cannot account for the large variation in input torque required to maintain constant crank speed. What then is the explanation? The large variations in torque are evidence of the kinetic energy that is stored in the links as they move. We can think of the positive pulses of torque as representing energy delivered by the driver (motor) and stored temporarily in the moving links, and the negative pulses of torque as energy attempting to return from the links to the driver. Unfortunately most motors are designed to deliver energy but not to take it back. Thus the "returned energy" has no place to go.

Figure 11-9 shows the speed torque characteristic of a permanent magnet (PM) DC electric motor. Other types of motors will have differently shaped functions that relate motor speed to torque as shown in Figure 2-32 and 2-33 (pp. 62-63), but all drivers



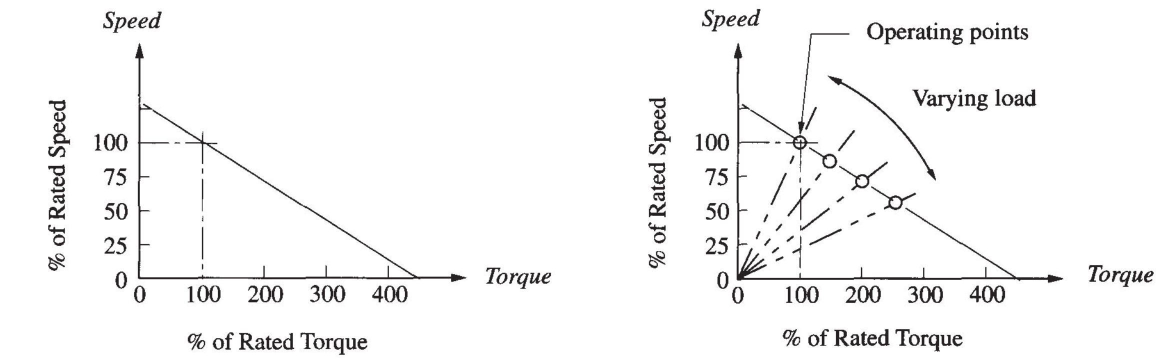
Link No.	Length in	Mass Units	Inertia Units	CG Posit	at Deg	Ext. Force lb	at Deg
1	5.5						
2	2.0	.002	.004	1.0	0		
3	6.0	.030	.060	2.5	30	12	270
4	3.0	.010	.020	1.5	0	60	–45

Coupler pt. = 3 in @ 45°
Open/Crossed = open

Ext. Force 3 acts at 5 in @ 30° vs. CG of Link 3
Ext. Force 4 acts at 5 in @ 90° vs. CG of Link 4
Ext. Torque 3 = – 20 lb-in
Ext. Torque 4 = 25 lb-in
Start Alpha2 = 0 rad/sec^2
Start Omega2 = 50 rad/sec
Start Theta2 = 0°
Final Theta2 = 360°
Delta Theta2 = 2°

FIGURE 11-8

Linkage data and input torque curve for an unbalanced crank-rocker fourbar linkage from program FOURBAR



(*a*) Speed–torque characteristic of a PM electric motor (*b*) Load lines superposed on speed–torque curve

FIGURE 11-9

DC permanent magnet (PM) electric motor's typical speed-torque characteristic

(sources) will have some such characteristic curve. As the torque demands on the motor change, the motor's speed must also change according to its inherent characteristic. This means that the torque curve being demanded in Figure 11-8 will be very difficult for a standard motor to deliver without drastic changes in its speed.

The computation of the torque curve in Figure 11-8 was made on the assumption that the crank (thus the motor) speed was a constant value. All the kinematic data used in the force and torque calculation was generated on that basis. With the torque variation shown we would have to use a large-horsepower motor to provide the power required to reach that peak torque at the design speed:

$$\begin{aligned} Power &= torque \times angular\ velocity \\ Peak\ power &= 341.7\ \text{lb-in} \times 50\,\frac{\text{rad}}{\text{sec}} = 17{,}085\,\frac{\text{in-lb}}{\text{sec}} = 2.59\ \text{hp} \end{aligned}$$

The power needed to supply the average torque is much smaller.

$$Average\ power = 70.2\ \text{lb-in} \times 50\,\frac{\text{rad}}{\text{sec}} = 3{,}510\,\frac{\text{in-lb}}{\text{sec}} = 0.53\ \text{hp}$$

It would be extremely inefficient to specify a motor based on the peak demand of the system, as most of the time it will be underutilized. We need something in the system which is capable of storing kinetic energy. One such kinetic energy storage device is called a **flywheel**.

FLYWHEEL ENERGY Figure 11-10 shows a **flywheel**, designed as a flat circular disk, attached to a motor shaft which might also be the driveshaft for the crank of our linkage. The motor supplies a torque magnitude T_M which we would like to be as constant as possible, i.e., to be equal to the average torque T_{avg}. The load (our linkage), on the other side of the flywheel, demands a torque T_L which is time varying as shown in Figure 11-8. The kinetic energy in a rotating system is:

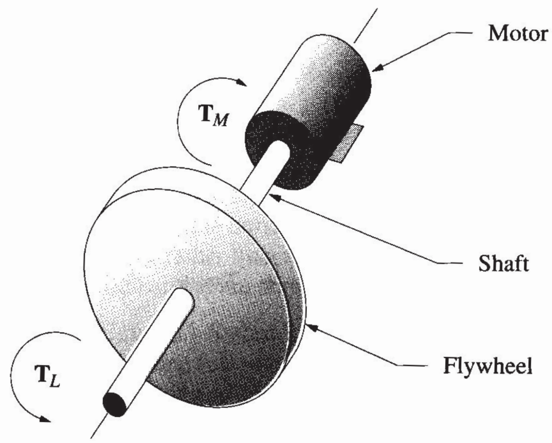


FIGURE 11-10
Flywheel on a driveshaft

$$E = \frac{1}{2} I \omega^2 \tag{11.17}$$

where I is the moment of inertia of all rotating mass on the shaft. This includes the I of the motor rotor and of the linkage crank plus that of the flywheel. We want to determine how much I we need to add in the form of a flywheel to reduce the speed variation of the shaft to an acceptable level. We begin by writing Newton's law for the free-body diagram in Figure 11-10.

$$\sum T = I\alpha$$

$$T_L - T_M = I\alpha$$

but we want:

$$T_M = T_{avg}$$

so:

$$T_L - T_{avg} = I\alpha \tag{11.18a}$$

substituting:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{dt}\left(\frac{d\theta}{d\theta}\right) = \omega \frac{d\omega}{d\theta}$$

gives:

$$T_L - T_{avg} = I\omega \frac{d\omega}{d\theta}$$

$$\left(T_L - T_{avg}\right) d\theta = I\omega \, d\omega \tag{11.18b}$$

and integrating:

$$\int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} \left(T_L - T_{avg}\right) d\theta = \int_{\omega_{min}}^{\omega_{max}} I\omega \, d\omega$$

$$\int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} \left(T_L - T_{avg}\right) d\theta = \frac{1}{2} I \left(\omega_{max}^2 - \omega_{min}^2\right) \tag{11.18c}$$

The left side of this expression represents the change in energy E between the maximum and minimum shaft ω's and is equal to the *area under the torque-time diagram** (Figures 11-8, p. 548, and 11-11) between those extreme values of ω. The right side of equation 11.18c is the change in energy stored in the flywheel. The only way we can extract energy from the flywheel is to slow it down as shown in equation 11.17. Adding energy will speed it up. Thus it is impossible to obtain exactly constant shaft velocity in the face of changing energy demands by the load. The best we can do is to minimize the speed variation ($\omega_{max} - \omega_{min}$) by providing a flywheel with sufficiently large I.

EXAMPLE 11-5

Determining the Energy Variation in a Torque-Time Function.

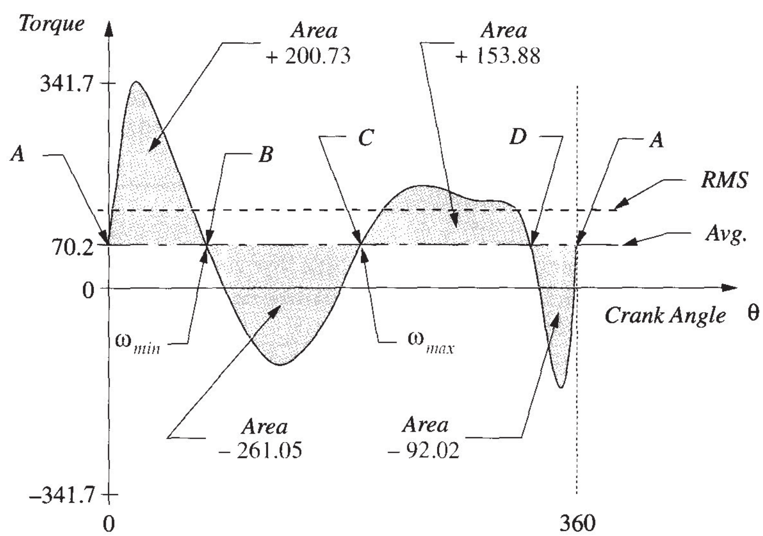
Given: An input torque-time function which varies over its cycle. Figure 11-11 shows the input torque curve from Figure 11-8. The torque is varying during the 360° cycle about its average value.

Find: The total energy variation over one cycle.

Solution:

1 Calculate the average value of the torque-time function over one cycle, which in this case is 70.2 lb-in. (Note that in some cases the average value may be zero.)

2 Note that the *integration on the left side of equation 11.18c is done with respect to the average line of the torque function, not with respect to the θ axis.* (From the definition of the



Areas of torque pulses in order over one cycle

Order	Neg Area	Pos Area
1	- 261.05	200.73
2	- 92.02	153.88

Energy units are lb-in-rad

FIGURE 11-11

Integrating the pulses above and below the average value in the input torque function

* There is often confusion between torque and energy because they appear to have the same units of *lb-in (in-lb)* or *N-m (m-N)*. This leads some students to think that they are the same quantity, but they are not. Torque ≠ energy. The **integral** of torque with respect to angle, measured in radians, **is** equal to energy. This integral has the units of *in-lb-rad*. The radian term is usually omitted since it is in fact unity. Power in a rotating system is equal to torque x angular velocity (measured in *rad/sec*), and the power units are then *(in-lb-rad)/sec*. When power is integrated versus time to get energy, the resulting units are *in-lb-rad*, the same as the integral of torque versus angle. The radians are again usually dropped, contributing to the confusion.

average, the sum of positive area above an average line is equal to the sum of negative area below that line.) The integration limits in equation 11.18 are from the shaft angle 8 at which the shaft (j)is a minimum to the shaft angle 8 at which (j)is a maximum.

3 The minimum (j)will occur after the maximum positive energy has been delivered from the motor to the load, i.e., at a point (8) where the summation of positive energy (area) in the torque pulses is at its largest positive value.

4 The maximum (j)will occur after the maximum negative energy has been returned to the load, i.e., at a point (8) where the summation of energy (area) in the torque pulses is at its largest negative value.

5 To find these locations in 8 corresponding to the maximum and minimum (j)'s and thus find the amount of energy needed to be stored in the flywheel, we need to numerically integrate each pulse of this function from crossover to crossover with the average line. The crossover points in Figure 11-11 have been labeled *A, B,* C, and D. (Program FOURBARdoes this integration for you numerically, using a trapezoidal rule.)

6 The FOURBARprogram prints the table of areas shown in Figure 11-11. The positive and negative pulses are separately integrated as described above. Reference to the plot of the torque function will indicate whether a positive or negative pulse is the first encountered in a particular case. The first pulse in this example is a positive one.

7 The remaining task is to accumulate these pulse areas beginning at an arbitrary crossover (in this case point *A)* and proceeding pulse by pulse across the cycle. Table 11-1 shows this process and the result.

8 Note in Table 11-1 that the minimum shaft speed occurs after the largest accumulated positive energy pulse (+200.73 in-lb) has been delivered from the driveshaft to the system. This delivery of energy slows the motor down. The maximum shaft speed occurs after the largest accumulated negative energy pulse (-60.32 in-lb) has been received back from the system by the driveshaft. This return of stored energy will speed up the motor. The total energy variation is the algebraic difference between these two extreme values, which in this example is -261.05 in-lb. This negative energy coming out of the system needs to be absorbed by the flywheel and then returned to the system *during each cycle* to smooth the variations in shaft speed.

TABLE 11-1 Integrating the Torque Function

From	Δ Area = ΔE	Accum. Sum = E	
A to *B*	+200.73	+200.73	ω_{min} @ *B*
B to *C*	−261.05	−60.32	ω_{max} @ *C*
C to *D*	+153.88	+93.56	
D to *A*	−92.02	+1.54	
	Total Δ Energy	$= E @ \omega_{max} - E @ \omega_{min}$	
		$= (-60.32) - (+200.73) = -261.05$ in-lb	

SIZING THE FLYWHEEL We now must determine how large a flywheel is needed to absorb this energy with an acceptable change in speed. The change in shaft speed during a cycle is called its *fluctuation (Fl)* and is equal to:

$$Fl = \omega_{max} - \omega_{min} \tag{11.19a}$$

We can normalize this to a dimensionless ratio by dividing it by the average shaft speed. This ratio is called the *coefficient of fluctuation (k)*.

$$k = \frac{(\omega_{\max} - \omega_{\min})}{\omega_{avg}} \tag{11.19b}$$

This *coefficient of fluctuation* is a design parameter to be chosen by the designer. It typically is set to a value between 0.01 and 0.05, which correspond to a 1 to 5% fluctuation in shaft speed. The smaller this chosen value, the larger the flywheel will have to be. This presents a design trade-off. A larger flywheel will add more cost and weight to the system, which factors have to be weighed against the smoothness of operation desired.

We found the required change in energy E by integrating the torque curve

$$\int_{\theta\,@\,\omega_{min}}^{\theta\,@\,\omega_{max}} \left(T_L - T_{avg}\right) d\theta = E \tag{11.20a}$$

and can now set it equal to the right side of equation 11.18c (p. 550):

$$E = \frac{1}{2} I \left(\omega_{max}^2 - \omega_{min}^2\right) \tag{11.20b}$$

Factoring this expression:

$$E = \frac{1}{2} I \left(\omega_{max} + \omega_{min}\right)\left(\omega_{max} - \omega_{min}\right) \tag{11.20c}$$

If the torque-time function were a pure harmonic, then its average value could be expressed exactly as:

$$\omega_{avg} = \frac{(\omega_{max} + \omega_{min})}{2} \tag{11.21}$$

Our torque functions will seldom be pure harmonics, but the error introduced by using this expression as an approximation of the average is acceptably small. We can now substitute equations 11.19b and 11.21 into equation 11.20c to get an expression for the mass moment of inertia I_s of the system flywheel needed.

$$E = \frac{1}{2} I \left(2\omega_{avg}\right)\left(k\,\omega_{avg}\right)$$

$$I_s = \frac{E}{k\,\omega_{avg}^2} \tag{11.22}$$

Equation 11.22 can be used to design the physical flywheel by choosing a desired coefficient of fluctuation k, and using the value of E from the numerical integration of

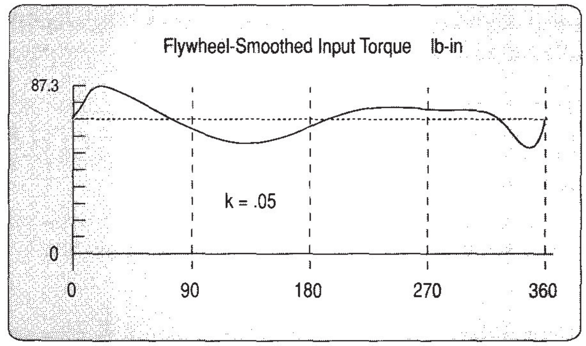


FIGURE 11-12

Input torque curve for the fourbar linkage in Figure 11-8 after smoothing with a flywheel

the torque curve (see Table 11-1, p. 552) and the average shaft (i) to compute the needed system I_s. The physical flywheel's mass moment of inertia I_f is then set equal to the required system I_s. But if the moments of inertia of the other rotating elements on the same driveshaft (such as the motor) are known, the physical flywheel's required I_f can be reduced by those amounts.

The most efficient flywheel design in terms of maximizing I_f for minimum material used is one in which the mass is concentrated in its rim and its hub is supported on spokes, like a carriage wheel. This puts the majority of the mass at the largest radius possible and minimizes the weight for a given I_f Even if a flat, solid circular disk flywheel design is chosen, either for simplicity of manufacture or to obtain a flat surface for other functions (such as an automobile clutch), the design should be done with an eye to reducing weight and thus cost. Since in general, I = m,.2, a thin disk of large diameter will need fewer pounds of material to obtain a given I than will a thicker disk of smaller diameter. Dense materials such as cast iron and steel are the obvious choices for a flywheel. Aluminum is seldom used. Though many metals (lead, gold, silver, platinum) are more dense than iron and steel, one can seldom get the accounting department's approval to use them in a flywheel.

Figure 11-12 shows the change in the input torque T_{12} for the linkage in Figure 11-8 after the addition of a flywheel sized to provide a coefficient of fluctuation of 0.05. The oscillation in torque about the unchanged average value is now 5%, much less than what it was without the flywheel. A much smaller horsepower motor can now be used because the flywheel is available to absorb the energy returned from the linkage during its cycle.

11.12 A LINKAGE FORCE TRANSMISSION INDEX

The transmission angle was introduced in Chapter 2 and used in subsequent chapters as an index of merit to predict the kinematic behavior of a linkage. A too-small transmission angle predicts problems with motion and force transmission in a fourbar linkage. Unfortunately, the transmission angle has limited application. It is only useful for fourbar linkages and then only when the input and output torques are applied to links that are pivoted to ground (i.e., the crank and rocker). When external forces are applied to the coupler link, the transmission angle tells nothing about the linkage's behavior.

Holte and Chase [1] define a joint-force index *(IPl)* which is useful as an indicator of any linkage's ability to smoothly transmit energy regardless of where the loads are applied on the linkage. It is applicable to higher-order linkages as well as to the fourbar linkage. The IPI at any instantaneous position is defined as the ratio of the maximum static force in any joint of the mechanism to the applied external load. If the external load is a force, then it is:

$$JFI = MAX\left|\frac{F_{ij}}{F_{ext}}\right| \qquad \text{for all pairs } i, j \qquad (11.23a)$$

If the external load is a torque, then it is:

$$JFI = MAX\left|\frac{F_{ij}}{T_{ext}}\right| \qquad \text{for all pairs } i, j \qquad (11.23b)$$

where, in both cases, F_{ij} is the force in the linkage joint connecting links i and j.

The Fij are calculated from a static force analysis of the linkage. Dynamic forces can be much greater than static forces if speeds are high. However, if this static force transmission index indicates a problem in the absence of any dynamic forces, then the situation will obviously be worse at speed. The largest joint force at each position is used rather than a composite or average value on the assumption that high friction in anyone joint is sufficient to hamper linkage performance regardless of the forces at other joints.

Equation 11.23a is dimensionless and so can be used to compare linkages of different design and geometry. Equation 11.23b has dimensions of reciprocal length, so caution must be exercised when comparing designs when the external load is a torque. Then the units used in any comparison must be the same, and the compared linkages should be similar in size.

Equations 11.23 apply to anyone instantaneous position of the linkage. As with the transmission angle, this index must be evaluated for all positions of the linkage over its expected range of motion and the largest value of that set found. The peak force may move from pin to pin as the linkage rotates. If the external loads vary with linkage position, they can be accounted for in the calculation.

Holte and Chase suggest that the IPI be kept below a value of about 2 for linkages whose output is a force. Larger values may be tolerable especially if the joints are designed with good bearings that are able to handle the higher loads.

There are some linkage positions in which the IPI can become infinite or indeterminate as when the linkage reaches an immovable position, defined as the input link or input joint being inactive. This is equivalent to a stationary configuration as described in earlier chapters provided that the input joint is inactive in the particular stationary configuration. These positions need to be identified and avoided in any event, independent of the determination of any index of merit. In some cases the mechanism may be immovable but still capable of supporting a load. See reference [1] for more detailed information on these special cases.

11.13 PRACTICAL CONSIDERATIONS

This chapter has presented some approaches to the computation of dynamic forces in moving machinery. The newtonian approach gives the most information and is necessary in order to obtain the forces at all pin joints so that stress analyses of the members can be done. Its application is really quite straightforward, requiring only the creation of correct free-body diagrams for each member and the application of the two simple vector equations which express Newton's second law to each free-body. Once these equations are expanded for each member in the system and placed in standard matrix form, their solution (with a computer) is a trivial task.

The real work in designing these mechanisms comes in the determination of the shapes and sizes of the members. In addition to the kinematic data, the force computation requires only the masses, CG locations, and mass moments of inertia versus those *CGs* for its completion. These three geometric parameters completely characterize the member for dynamic modelling purposes. Even if the link shapes and materials are completely defined at the outset of the force analysis process (as with the redesign of an existing system), it is a tedious exercise to calculate the dynamic properties of complicated shapes. Current solids modelling *CAD* systems make this step easy by computing these parameters automatically for any part designed within them.

If, however, you are starting from scratch with your design, the *blank-paper syndrome* will inevitably rear its ugly head. A first approximation of link shapes and selection of materials must be made in order to create the dynamic parameters needed for a "first pass" force analysis. A stress analysis of those parts, based on the calculated dynamic forces, will invariably find problems that require changes to the part shapes, thus requiring recalculation of the dynamic properties and recomputation of the dynamic forces and stresses. This process will have to be repeated in circular fashion *(iteration-see* Chapter 1, p. 8) until an acceptable design is reached. The advantages of using a computer to do these repetitive calculations is obvious and cannot be overstressed. An equation solver program such as *TKSolver* or *Mathcad* will be a useful aid in this process by reducing the amount of computer programming necessary.

Students with no design experience are often not sure how to approach this process of designing parts for dynamic applications. The following suggestions are offered to get you started. As you gain experience, you will develop your own approach.

It is often useful to create complex shapes from a combination of simple shapes, at least for first approximation dynamic models. For example, a link could be considered to be made up of a hollow cylinder at each pivot end, connected by a rectangular prism along the line of centers. It is easy to calculate the dynamic parameters for each of these simple shapes and then combine them. The steps would be as follows (repeated for each link):

1 Calculate the volume, mass, CG location, and mass moments of inertia with respect to the local CG of each separate part of your built-up link. In our example link these parts would be the two hollow cylinders and the rectangular prism.

2 Find the location of the composite CG of the assembly of the parts into the link by the method shown in Section 11.4 (p. 531) and equation 11.3 (p. 524). See also Figure 11-2 (p. 526).

3 Use the *parallel axis theorem* (equation 10.8, p. 497) to transfer the mass moments of inertia of each part to the common, composite *CG* for the link. Then add the individual, transferred *I*'s of the parts to get the total *I* of the link about its composite *CG*. See Section 11.6 (p. 541).

Steps 1 to 3 will create the link geometry data for each link needed for the dynamic force analysis as derived in this chapter.

4 Do the dynamic force analysis.

5 Do a dynamic stress and deflection analysis of all parts.

6 Redesign the parts and repeat steps 1 to 5 until a satisfactory result is achieved.

Remember that lighter (lower mass) links will have smaller inertial forces on them and thus could have lower stresses despite their smaller cross sections. Also, smaller mass moments of inertia of the links can reduce the driving torque requirements, especially at higher speeds. But be cautious about the dynamic deflections of thin, light links becoming too large. We are assuming rigid bodies in these analyses. That assumption will not be valid if the links are too flexible. Always check the deflections as well as the stresses in your designs.

11.14 REFERENCES

1 **Holte, J. E., and T. R. Chase**. (1994). "A Force Transmission Index for Planar Linkage Mechanisms." *Proc. of 23rd Biennial Mechanisms Conference*, Minneapolis, MN, p. 377.

11.15 PROBLEMS

11-1 Draw free-body diagrams of the links in the geared fivebar linkage shown in Figure 4-11 (p. 165) and write the dynamic equations to solve for all forces plus the driving torque. Assemble the symbolic equations in matrix form for solution.

11-2 Draw free-body diagrams of the links in the sixbar linkage shown in Figure 4-12 (p. 167) and write the dynamic equations to solve for all forces plus the driving torque. Assemble the symbolic equations in matrix form for solution.

*†‡11-3 Table P11-1 shows kinematic and geometric data for several slider-crank linkages of the type and orientation shown in Figure P11-1. The point locations are defined as described in the text. For the row(s) in the table assigned, use the matrix method of Section 11.5 (p. 538) and program MATRIX, *Mathcad*, *Matlab*, *TKSolver*, or a matrix solving calculator to solve for forces and torques at the position shown. Also compute the shaking force and shaking torque. Consider the coefficient of friction μ between slider and ground to be zero. You may check your solution by opening the solution files (located in the Solutions folder on the CD-ROM) named P11-03x (where x is the row letter) into program SLIDER.

*†11-4 Repeat Problem 11-3 using the method of virtual work to solve for the input torque on link 2. Additional data for corresponding rows are given in Table P11-2.

*†11-5 Table P11-3 shows kinematic and geometric data for several pin-jointed fourbar linkages of the type and orientation shown in Figure P11-2. All have $\theta_1 = 0$. The

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

§ These problems are suited to solution using program SLIDER which is on the attached CD-ROM.