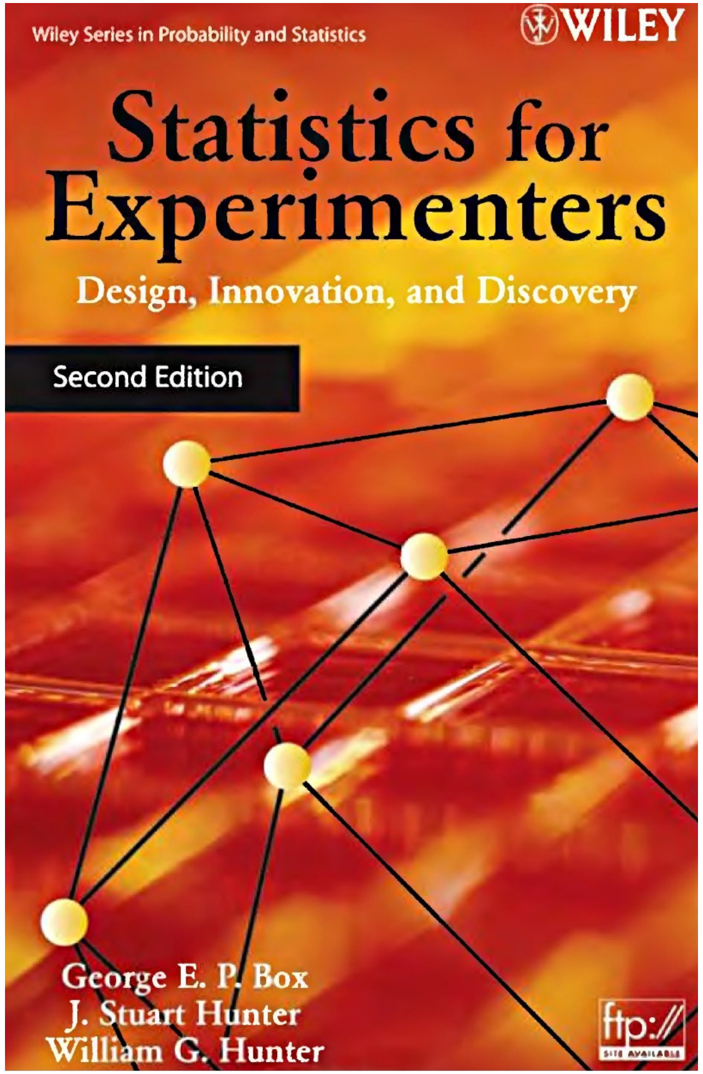
Powertrain Calibration Optimisation

Introduction to Statistics

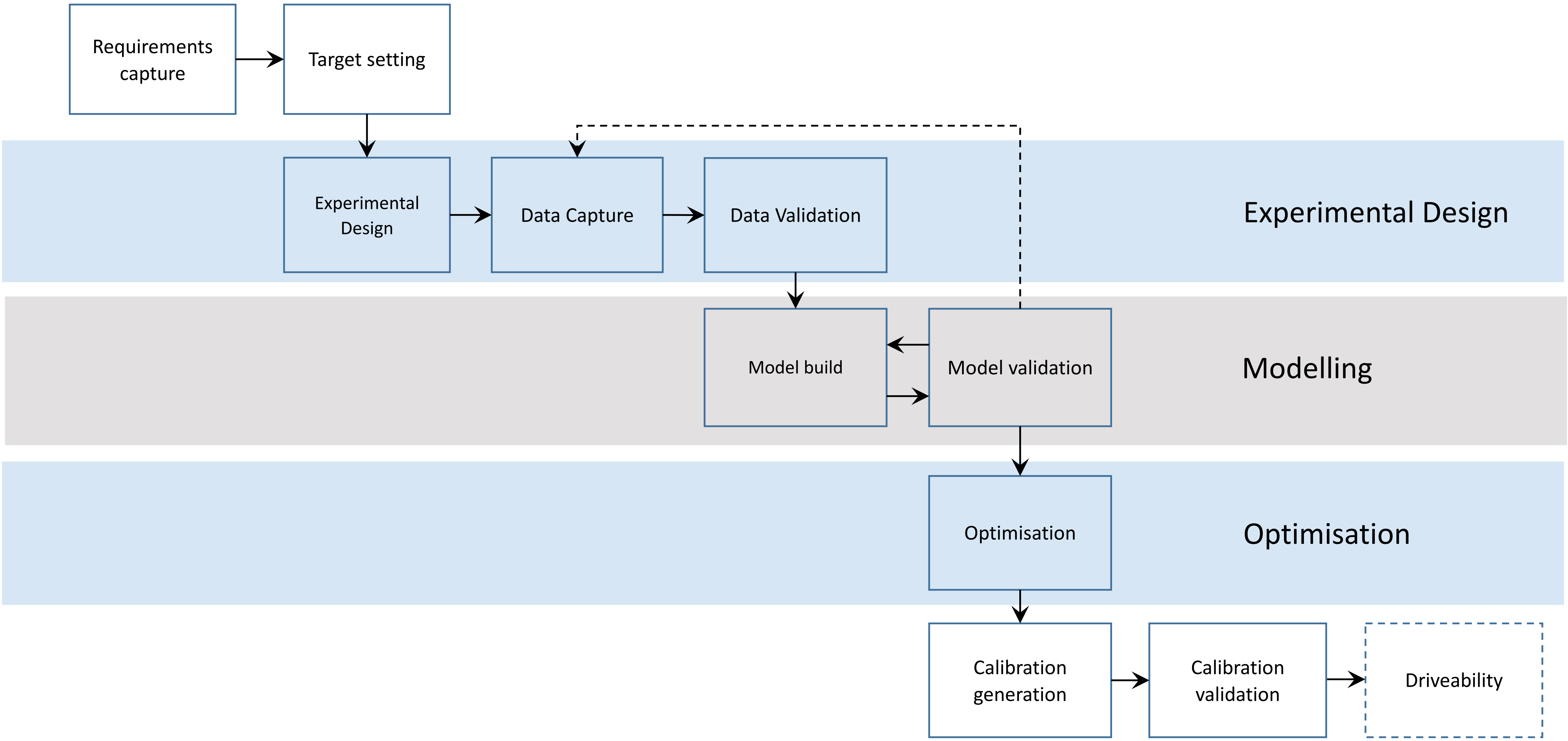
Overview

- Process overview
- Basic concepts
- Continuous distributions
- Estimation
- Significance tests
- Regression
- ANOVA

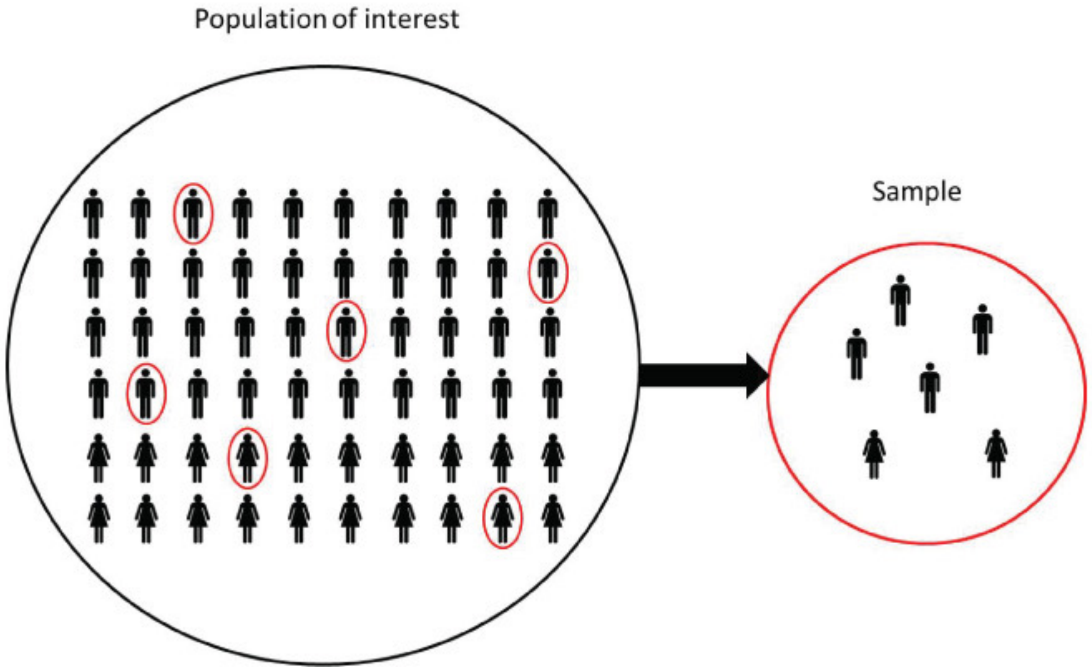

Wiley Series in Probability and Statistics
WILEY
Statistics for Experimenters
Design, Innovation, and Discovery
Second Edition
George E. P. Box
J. Stuart Hunter
William G. Hunter
ftp://

Powertrain Calibration Optimisation

High Level Overview



Population vs sample



As the sample size, n increases, the sample becomes more representative of the population from which it is drawn.

Definition - Degrees of Freedom

- How many choices?
- Degrees of freedom* relate to the number of ‘observations’ that are free to vary when estimating statistical parameters

$$mean = \frac{x_1 + x_2 + \cdots x_n}{n}$$

In calculating the mean only $n-1$ observations are ‘free to vary’

* we also talk about control degrees of freedom which is the control inputs that we can change to modify the behaviour of the system.

Making measurements – location and spread

Mean, $\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N} = \sum_N \frac{x_i}{N}$

Also known as the expectation of x i.e. $E(x)$.

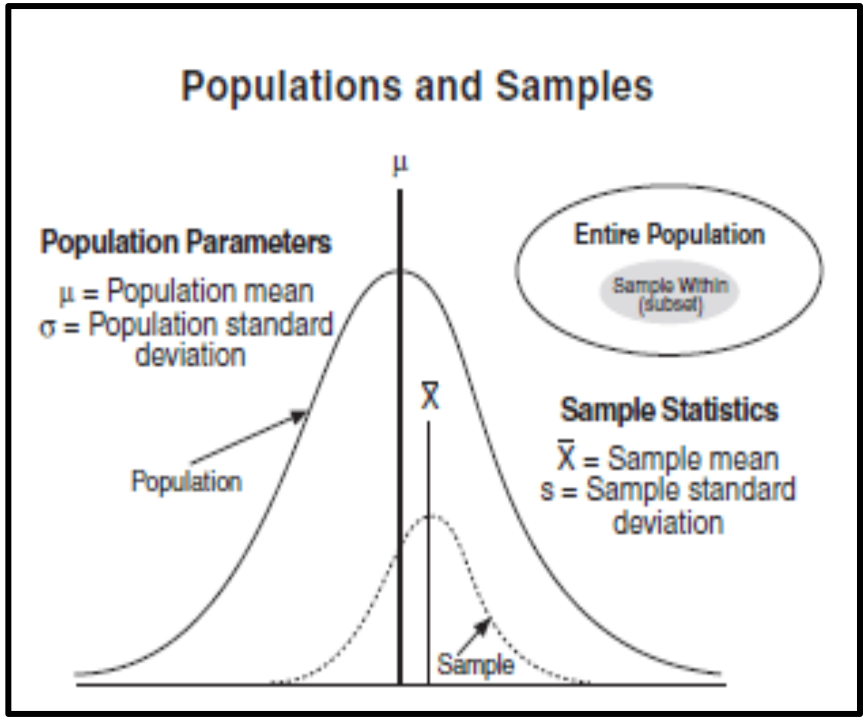
Variance, s^2

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2}{N-1} = \sum_N \frac{(x_i - \bar{x})^2}{N-1}$$

Why $N - 1$?

Standard deviation, s,

$$s = \sqrt{s^2}$$



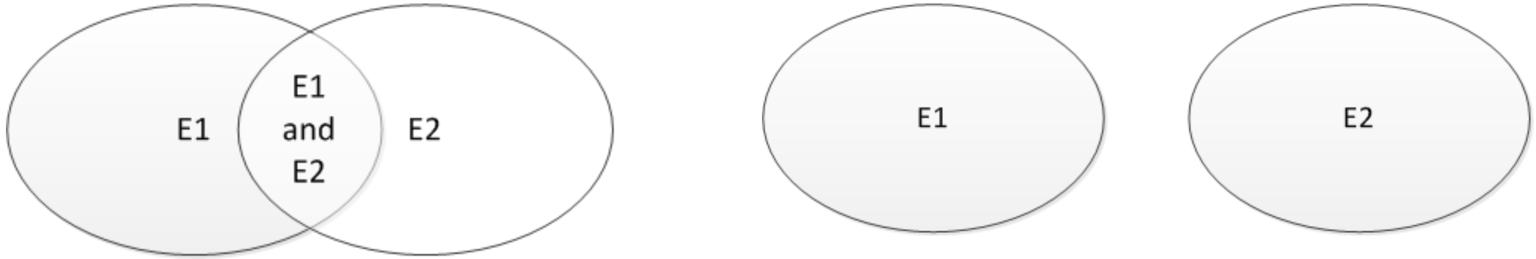
What are the units of s, s^2, σ, σ^2?

Probability

Probability: *the extent to which an event is likely to occur, measured by the ratio of the cases of interest to the whole number of cases possible*

Sample space: *the set of all possible outcomes of an experiment.*

Events:

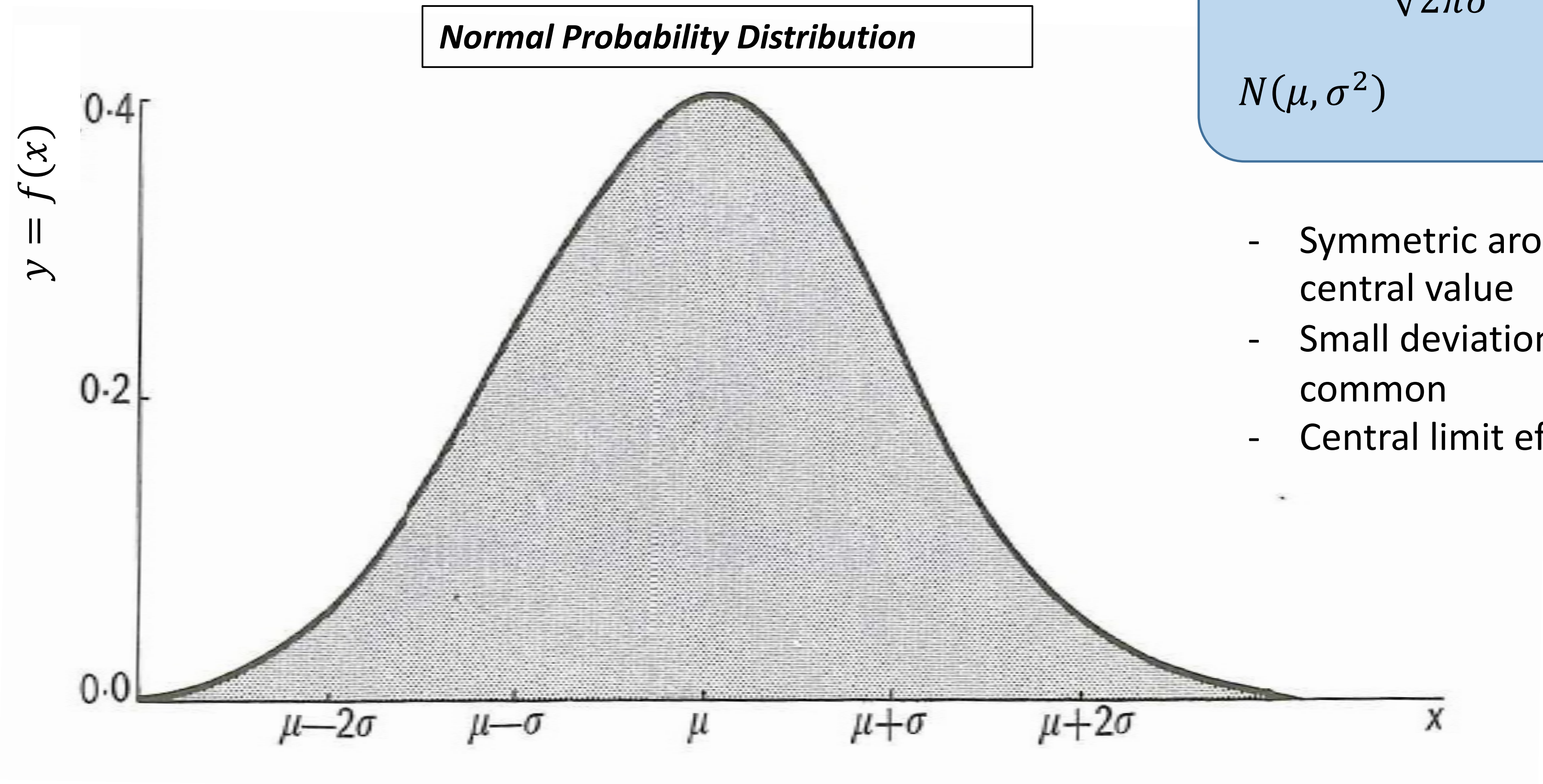


Probabilities: $p(E_1), p(E_2), p(E_1E_2)$

$$P(E_1|E_2) = \frac{\sum_{E_1E_2} P(Sample\ points\ common\ to\ E_1 and\ E_2)}{\sum_{E_2} P(Sample\ points\ in\ E_2)} = \frac{P(E_1E_2)}{P(E_2)}$$

"E_2 has already happened." What is the probability of E1?

Probability distributions

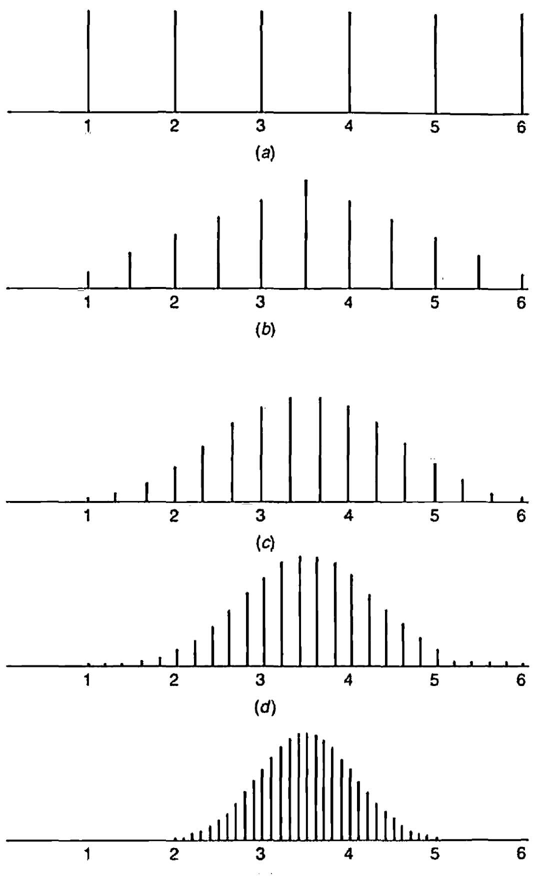


$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2 / 2\sigma^2}$$

$$N(\mu, \sigma^2)$$

- Symmetric around some central value
- Small deviations more common
- Central limit effect

Central limit effect



Average scores of (100 rolls)

One die

Two die

Three die

Five die

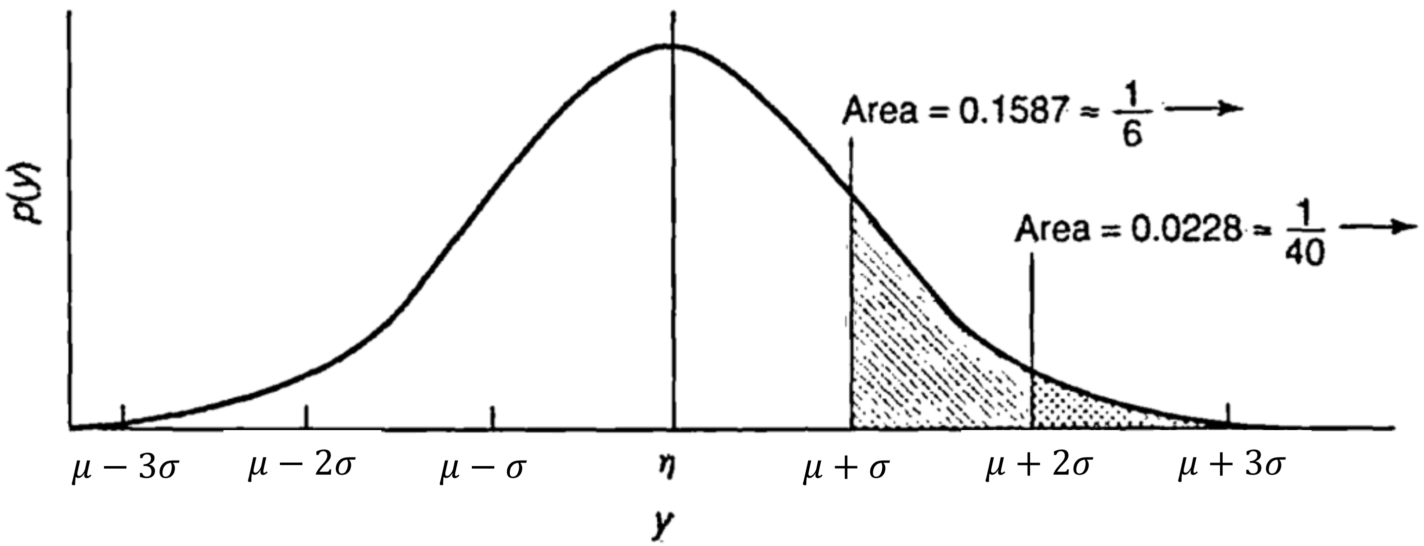
Ten die

In many experiments the error is an aggregate of a number of component errors and the distribution will tend to be "normal".

This is important since it is true for 'many' experiments.

Probability

μ and σ^2 fully characterise a normal distribution, $N(\mu, \sigma^2)$



Probability density is given by a point on the line p(y)

$p(y > \mu + \sigma) = \frac{1}{6}$ i.e. the area under the curve.

Often it is easier to express probability in terms of the standard deviate;

$$z = \frac{y - \mu}{\sigma}$$

$$z(0, 1)$$

i.e. z has a mean of 0 and variance, $s^2 = 1$. So that;

$$= p(y > \mu + \sigma)$$

$$= p(y - \mu > \sigma)$$

$$= p\left(\frac{y-\mu}{\sigma} > 1\right)$$

$$= p(z > 1)$$

(which can be easily found from tables)

If σ is unknown (which is normally the case)

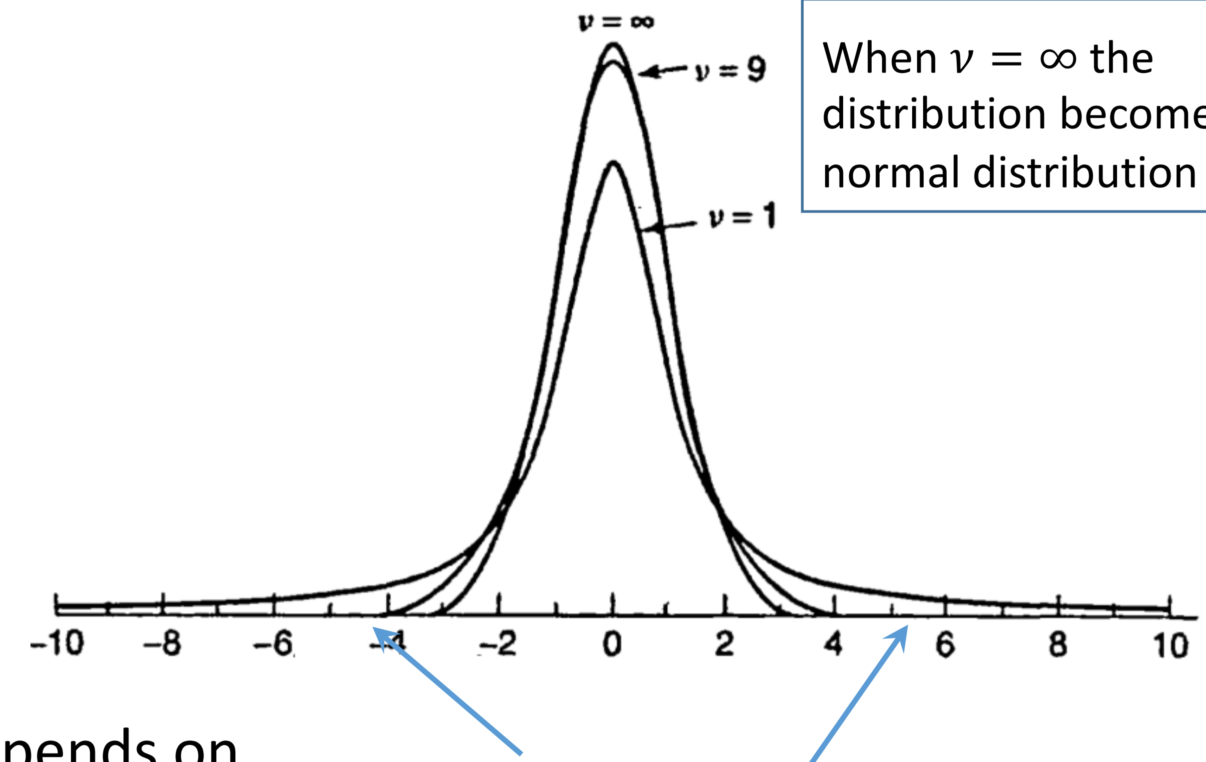
A substitution can be made for σ *using* s, the sample standard deviation;

$$z = \frac{y - \mu}{\sigma}$$

i.e.

$$t = \frac{y - \mu}{s}$$

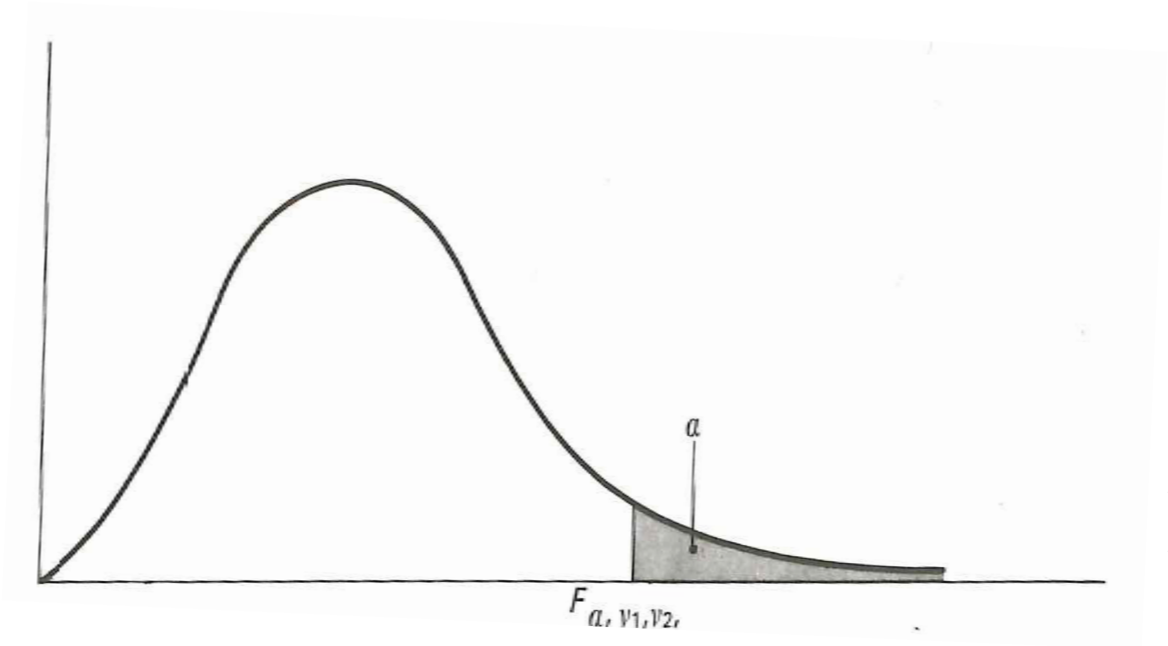
the ‘student’ or ‘t’ distribution depends on degrees of freedom available for estimation of s.



When $\nu = \infty$ the distribution becomes the normal distribution

Fatter tail compared with normal distribution

F-distribution



Can obtain ratio of two sample variances;

F statistic is ${s_1}^2/{s_2}^2$

F depends on the estimates and the DOF of the variance estimates

Degrees of freedom of population variances;

$\nu_1 = n_1 - 1$
$\nu_2 = n_2 - 1$

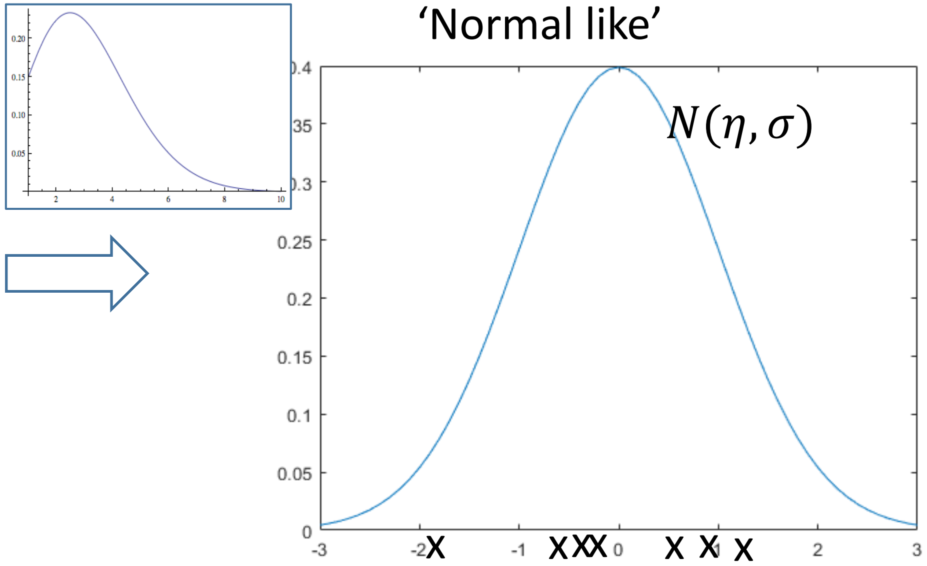
So F test statistic is designated

F_{ν_1, ν_2}

Standard Error of the Mean

- Take n random samples from a normal distribution with mean, μ and standard deviation, σ. Calculate the sample $\bar{x}$ and s. Repeat.
- The sample means will form a distribution with the same mean, μ but a **smaller standard deviation** $\sigma/\sqrt{n}$ (the **standard error of the sample mean**).

'Normal like'



For a sample of size n the sample mean is $\bar{x}$

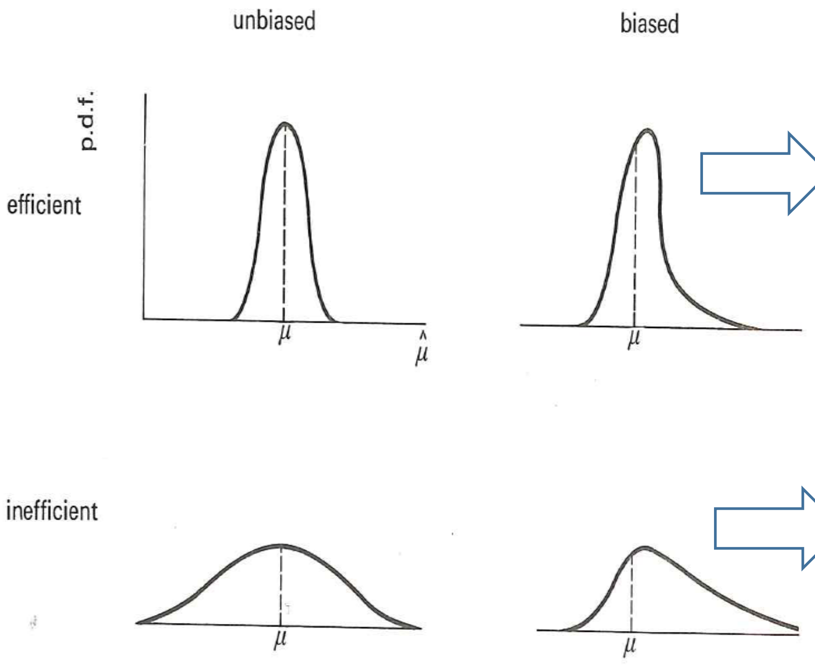
The standard error is an estimate of the standard deviation of the sample means for sample size n

$$SE_m = \frac{\sigma}{\sqrt{n}}$$

Intuitively it is a measure of how sample size affects the dispersion of sample means relative to the population mean.

Bias and efficiency

- **Bias** – an estimator is said to be biased if the mean of its sampling distribution is not equal to the value it is estimating.
- **Efficiency** – an efficient unbiased estimator is the minimum variance unbiased estimator (MVUE).



$bias(\hat{\mu}) = \mathrm{E}(\mu) - \mu$

$var(\hat{\mu}_1) < var(\hat{\mu}_2)$

more efficient estimator $\hat{\mu}_1$

Significance testing

Testing a theory about the population

Null hypothesis	H_0
Alternative hypothesis	H_1

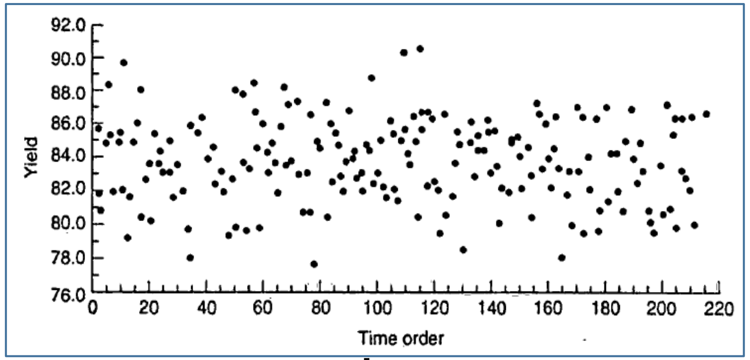
What we are testing ….

An alternative ….

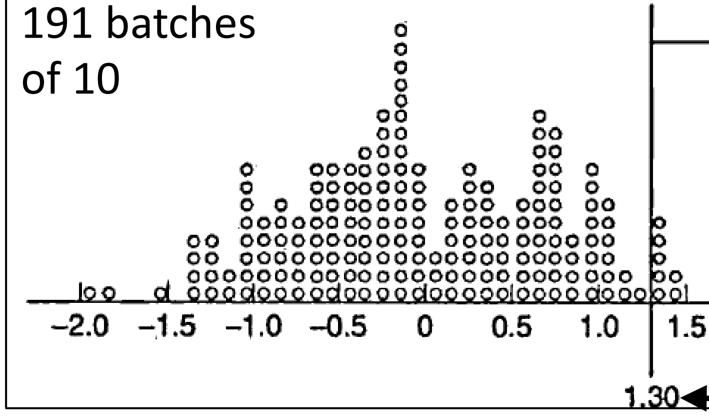
- Test statistic
- Level of significance
- One tailed and two tailed tests

How to know if a treatment is significant?

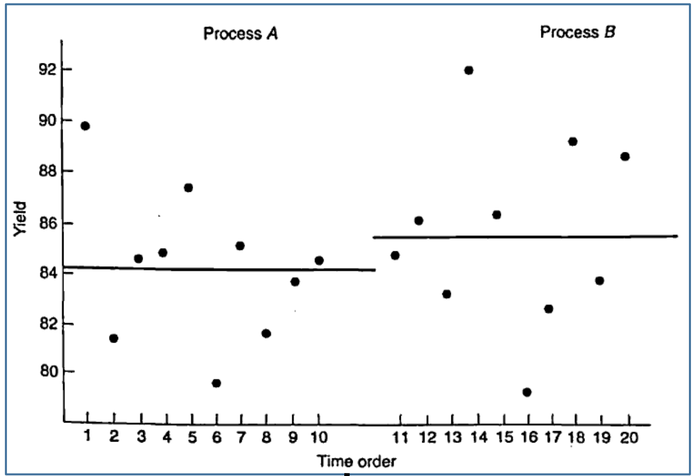
Previous yield data



191 batches of 10



$$\frac{9}{191} = 0.047$$



Difference in means of sequential batches

$\bar{y}_a = 84.24$

$\bar{y}_b = 85.54$

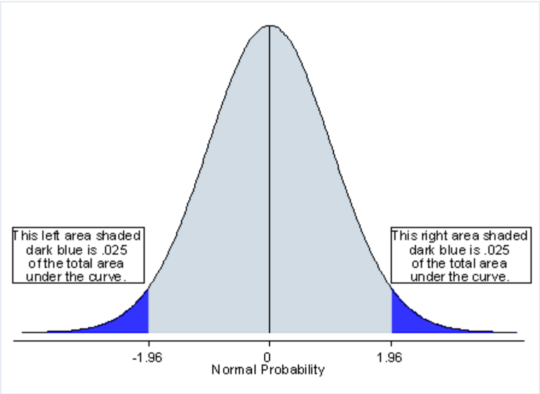
Difference 1.3

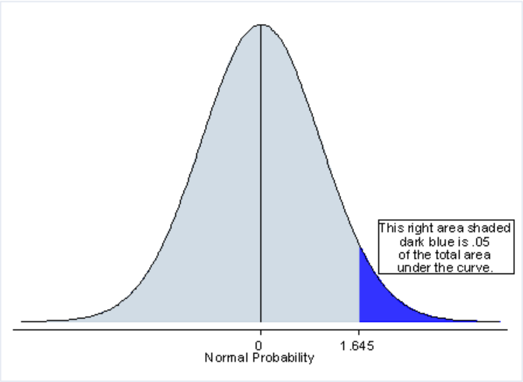
An example - composition of a chemical compound

- The iron content of a compound should be 12.1%. Tests on nine different samples are being used to examine this assumption.

- Null hypothesis i.e. there is no difference in the sample $(n = 9)$ mean
  - $H_0: \mu = 12.1\%$

- Alternative hypothesis
  - $H_1: \mu \neq 12.1\%$





Example (continued)

The analysis of nine samples gave the following values for % content of iron.

11.7	12.2	10.9	11.4	11.3	12.0	11.1	10.7	11.6

$$\bar{y} = 11.43$$
$$s^2 = 0.24$$
$$s = 0.49$$

$$t = \frac{(\bar{y} - \eta)}{s/\sqrt{n}} = \frac{(11.43 - 12.1)}{0.49/\sqrt{9}} = -4.1$$

Standard deviation of the mean (estimate)

Degrees of freedom: eight because nine samples and one DoF used for population mean, $\bar{x}$

We are trying to work out the probability that the differences in the means are of significance or not (relative to some acceptable level).

If they are we reject the null hypothesis.

Example (continued)

1. Two tailed test
2. 5% level of significance
3. Eight degrees of freedom (from tables)

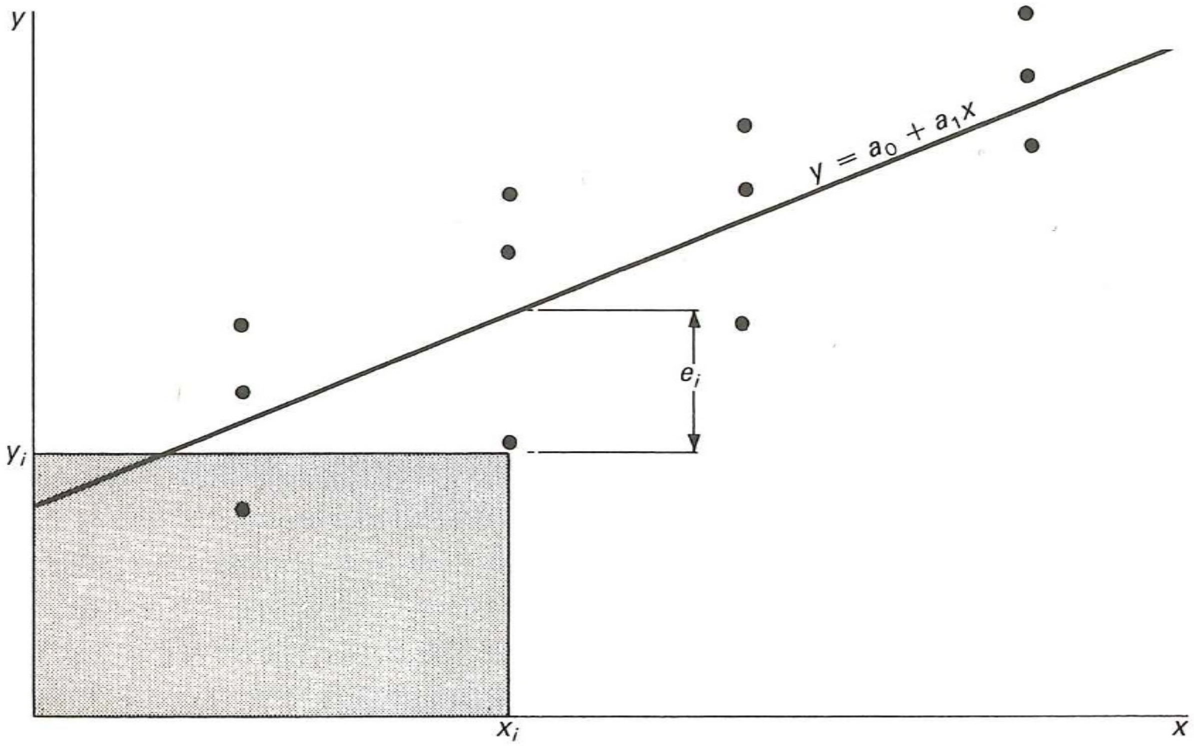
Test statistic is designated: $t_{0.025,8} = 2.31$ (from tables)

In fact, $t_{0.005,8} = 3.36$ (from tables)

So even at the 1% level, the result is significant.

Regression

Fitting a line or curve to the data in order to predict the mean value of the dependent variable for a given value of the controlled variable

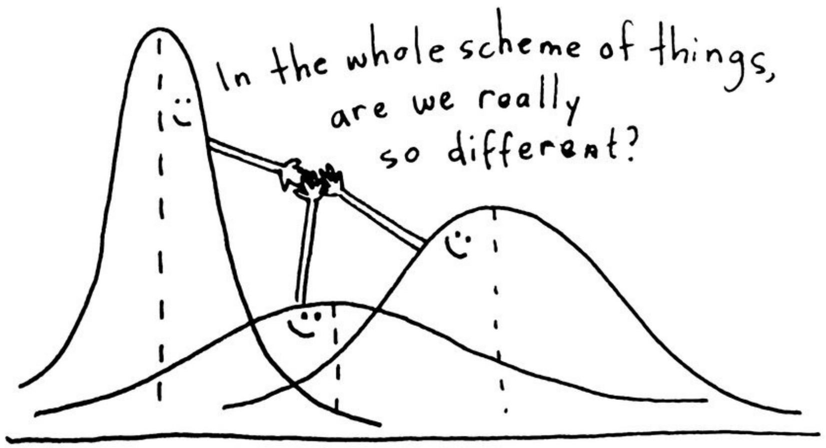


Model = $f(x, \beta)$

$$\min\left(\sum y_i - f(x_i, \beta)\right)$$

Analysing variance (ANOVA)
For comparing more than two entities

	A	B	C	D
	62	63	68	56
	60	67	66	62
	63	71	71	60
	59	64	67	61
	63	65	68	63
	59	66	68	64
Group avg	**61**	**66**	**68**	**61**
Overall avg	**64**	**64**	**64**	**64**



ANOVA

- Review the topic and evaluate the data on the previous slide.
- Prepare a 5 minute presentation for next week.
- Randomly chosen presenter (MATLAB script will make the choice)

Deviation from overall average

Deviations **between** treatments

deviations **within** treatments

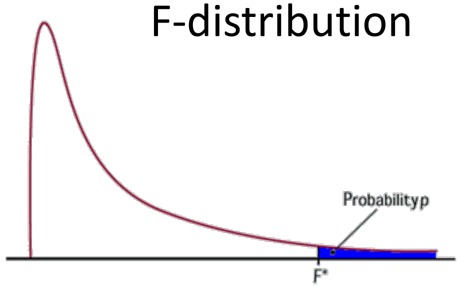
	A	B	C	D	$y_{ti} - \bar{y}$	$\bar{y}_t - \bar{y}$	$y_{ti} - \bar{y}_t$
y_{ti}	62	63	68	56	-2 -1 4 -8	-3 2 4 -3	1 -3 0 -5
	60	67	66	62	-4 3 2 -2	-3 2 4 -3	-1 1 -2 1
	63	71	71	60	-1 7 7 -4	-3 2 4 -3	2 5 3 -1
	59	64	67	61	-5 0 3 -3	-3 2 4 -3	-2 -2 -1 0
	63	65	68	63	-1 1 4 -1	-3 2 4 -3	2 -1 0 2
	59	66	68	64	5 2 4 0	-3 2 4 -3	-2 0 0 3
			Sum of squares		**340**	**228**	**112**
			Degrees of freedom		**23**	**3**	**20**

y_{ti} individual results
$\bar{y}_t$ treatment average
$\bar{y}$ overall average

ANOVA Table

Source of variation	Sum of squares	d.f.	$\frac{\chi^2}{\nu}$
Between treatments	$\sum(\bar{y}_t - \bar{\boldsymbol{y}})^2 = 228$	$n-1=3$	$\frac{\sum(\bar{y}_t - \bar{\boldsymbol{y}})^2}{n-1} = 76$
Within treatments	$\sum(\boldsymbol{y}_{ti} - \bar{\boldsymbol{y}}_t)^2 = 112$	$n-1=20$	$\frac{\sum(y_{ti} - \bar{y}_t)^2}{n-1} = 5.6$
Total about the overall average	**340**	**23**	

F-distribution



$$F_{\nu_1,\nu_2} = \frac{\sum(\bar{y}_t - \bar{y})^2}{n-1} \Big/ \frac{\sum(y_{ti} - \bar{y}_t)^2}{n-1}$$

$$F_{3,20} = 13.6$$

Significant at 0.001 i.e. we can be confident that treatments do result in different means, we can reject H_0